Computer Science 715 Spring 2012
Final Examination, May 12, 2012

1. The *degree* of a vertex of a graph is the number of neighbors of that vertex. The degree of a graph is the maximum degree of any vertex of the graph. Suppose that G is a connected graph with n vertices, whose degree and diameter are both d. Prove that $d = \Omega\left(\frac{\log n}{\log\log n}\right)$.

2. Solve the recurrence:

$$F(n) = \frac{n}{\log n} F(\log n) + n$$

3. Use amortized analysis to prove that the following pseudo-code takes $O(k)$ time to execute. The function UNKNOWN has Boolean type, and returns whatever the adversary decides each time it is called.

```
i = 0
for j = 1 to k do
  while i > 0 and UNKNOWN do
    i = i - 1
  end while
  i = i + 1
end for
```

4. Using the result from Problem 3, prove that the REDUCE subroutine of SMAWK takes $O(n+m)$ time to reduce the case of an $n \times m$ strictly monotone matrix, for $m \geq n$, to the case of an $n \times n$ strictly monotone matrix.

5. Give a solution to the range query problem of size n that takes $O(n \log^* n)$ preprocessing time and $O(1)$ time for each query, and which uses $O(n \log^* n)$ space.

6. Given a sequence of integers, it is possible to find the longest monotone increasing subsequence in $O(n \log n)$ time, where n is the length of the sequence. Explain how that is done.

   Walk through your algorithm for the example sequence:

   At each step, illustrate the current state of the data structure.

7. Explain Johnson's algorithm for the all-pairs shortest path problem on a sparse weighted directed graph of n nodes and m edges where there are no negative cycles (although there could be negative edges). Give the asymptotic time complexity in terms of n and m.

8. Prove that the monotone Boolean circuit problem is $\mathcal{P}$-complete, assuming that the Boolean circuit problem is $\mathcal{P}$-complete.

9. Prove that the tropical product of two Monge matrices is Monge.

10. Give an $O(n^2)$-time algorithm which computes the tropical product of two $n \times n$ Monge matrices.

11. Consider a $3 \times n$ weighted grid graph, G. Let s be the upper left corner node. Prove that there is a parallel algorithm in the CREW PRAM model that solves the single-source minpath problem in G, where the source node is s, with time complexity $O(\log n)$ using $\frac{n}{\log n}$ processors. You don't have to actually give the algorithm. You only have to convince me that the algorithm exists.

    Be careful. The least cost path from s to the opposite corner could be a path of length $3n + 1$ that uses every node.

12. Solve the following recurrences.

    (a) $F(n) = F(\frac{n}{2}) + 6F(\frac{n}{3}) + 3F(\frac{n}{6}) + n^2$

    (b) $F(n) = F(\frac{4n}{9}) + F(\frac{n}{9}) + 1$

    (c) $F(n) = F(\frac{4n}{9}) + F(\frac{n}{9}) + n$

13. The following 0-1 problems are all obviously in the class $\mathcal{NP}$.

    P1: SAT.

    P2: 3-SAT.

    P3: 3-in-1 SAT.

    P4: Independent Set.

    P5: Clique.

    P6: Knapsack.

    P7: Traveling Salesman.

    P8: Dominating Set.

    P9: Integer Programming.

    One of the following will be on the test:

    (a) Assuming that P1, P2, and P3 are $\mathcal{NP}$-complete, prove that P4 is $\mathcal{NP}$-complete.

    (b) Assuming that P1, P2, P3, P4, and P5 are $\mathcal{NP}$-complete, prove that P6 is $\mathcal{NP}$-complete.

    (c) Assuming that P1, P2, P3, P4, P5, P6 and P7 are $\mathcal{NP}$-complete, prove that P8 is $\mathcal{NP}$-complete.