The Complexity of Illuminating Polygons by α-Flood-Lights

Jay Bagga [1], Laxmi Gewali [2], and David Glasser [2]

Abstract: The Flood-light Illumination Problem (FIP) is the determination of the minimum number of vertex α-lights required to illuminate a given polygon. We show that FIP for simple polygon is NP-hard for any fixed α in the range $0 < \alpha \leq 360^0$. We further show that FIP for simple polygons is NP-Hard even when α-lights are required to be flush with the edges of the polygon. Our technique is based on the construction of beam-machines to reduce in polynomial time an instance of satisfiability problem to an instance of FIP.

1. Introduction

Study of the visibility properties of polygons is a rather popular research area in computational geometry [1-5]. An interesting visibility problem is the placement of the minimum number of point sources to illuminate the interior of a simple polygon. It may be noted that a point p inside a polygon is said to be illuminated by a point q (i.e., p and q are visible to each other) if the line segment connecting p and q does not intersect with the exterior of the polygon. In most illumination problems, visibility from a point is allowed in all directions (360^0 angular aperture). Recently, several researchers have considered illumination problems that restrict visibility to within a certain angular aperture [6,7,8]. A light source whose illumination angle is restricted to α-degrees is called an **α-flood-light** (or simply **α-light**) and an α-light placed on a vertex is a **vetex α-light**. It has been established that an orthogonal polygon of n sides can always be illuminated by $\lfloor \frac{3(n-1)}{8} \rfloor$ vertex 90^0-lights [6], which leads to a simple linear time placement algorithm. Illuminating non-orthogonal polygons by flood-lights seems to be a harder problem. A simple polygon may remain unilluminated even if we place one 90^0-light at each vertex; and this also holds true by restricting the polygon to be monotone [7]. Estivill-Castro, O'Rourke, Urrutia, and Xu [8] presented a surprising result on polygon illumination: there are simple polygons (called **logarithmic spirals**) that can not be illuminated by placing α-lights on all its vertices for any $\alpha \leq 180^0$. An open problem posed in [6] is the determination of the minimum number of α-lights required to illuminate a simple polygon. This problem is known to be NP-hard when visibility is allowed in all directions (i.e., $\alpha = 360^0$) [1,2]; the problem is also known to be NP-hard when $\alpha = 90^0$ [9].

[1] Department of Computer Science, Ball State University, Muncie, Indiana. email: jay@bsu-cs.bsu.edu
[2] Department of Computer Science, University of Nevada, Las Vegas, Nevada. e-mail: laxmi@cs.unlv.edu

In this paper, an approach based on beam-machines is used to show that the problem of illuminating a simple polygon by the minimum number of vertex α-lights is NP-hard for any α in range $0 < \alpha \leq 360^0$. We further show that the problem remains NP-hard (in fact NP-Complete) even if vertex α-lights are required to be flush along the edges of the polygon.

2. Preliminaries

The notation $< v_1, v_2, ..., v_n >$ is used to denote the **polygonal chain** connecting vertices in the order $v_1, v_2, v_3, ..., v_n$. A simple polygon is specified by the polygonal chain describing its boundary. In a polygonal chain, all vertices are distinct and hence the first vertex and the last vertex in the chain are not connected. However, when a polygonal chain $< v_1, v_2, ..., v_n >$ is used for representing a simple polygon then it is considered to be closed, i.e., the first vertex and the last vertex are understood to be connected. Our goal is to determine the complexity of illuminating a simple polygon by the minimum number of vertex α-lights. We require that at most one α-light can be placed on a vertex. By definition, the maximum angular aperture of an α-light is α degrees: this means that a vertex α-light may be used as a light source with any angular aperture β in the range $0 < \beta \leq \alpha$.

The Flood-light Illumination Problem (FIP)

Instance: *A simple polygon P of n sides, a positive integer m, and angular aperture α.*
Question: *Can P be illuminated by at most m vertex α-lights?*

The Satisfiability Problem (SAT)

Instance: *A collection W =$\{C_1, C_2, C_3, ..., C_k\}$ of clauses of a finite set of boolean variables $X = \{x_1, x_2, x_3, ..., x_r\}$; variable x_i occurs k_{i_1} times as x_i and k_{i_2} times as $\overline{x_i}$.*
Question: *Is there a truth assignment for variables in X that satisfies all the clauses?*

Given an instance I_1 of SAT, we convert it in polynomial time to an instance $I_2 = (P, m)$ of FIP such that I_1 is satisfiable if and only if the interior of P can be illuminated by m or fewer vertex α-lights, where m can be expressed in terms of the number of clauses k and the number of literals r in I_1. Our reduction is similar in spirit to the construction developed by Culberson and Reckhow [10], where an approach based on beam-machines is used to reduce the satisfiability problem to the problem of covering a simple polygon by the minimum number of convex polygons.

3. Illumination by 45^0-lights

For the purpose of clarity of exposition we first examine the complexity of FIP when the angular aperture is restricted to 45^0. One of the polygonal structures used in the reduction is the beam-machine.

3.1 Beam-Machine

A beam-machine is a polygonal chain used for generating a thin illumination beam by the placement of the minimum number of vertex 45^0-lights in its interior. Corresponding to each literal in the given SAT expression we will construct a beam-machine. Specifically, it is a nineteen sided polygonal chain $< a, b, c, ..., j, j', i', ...a' >$ (Figure 1a) which can be viewed as a simple polygon with a tiny opening $< a, a' >$ on its boundary. The interior of a beam-machine is considered to be the interior of the polygon formed by closing its opening. The interior of a beam-machine consist of four parts: (i) the **left-wing** $< d, e, f, g, h, i, j >$, (ii) the **right-wing** $< d', e', f', g', h', i', j' >$, (iii) the **body** (quadrilateral with vertices d, j, j', and d'), and (iv) the **mouth** $< d, c, b, a, a', b', c', d' >$. A beam-machine satisfies the visibility properties listed in Table 1.

Table 1

(*Visibility Properties of Beam-Machines*)

Pr.#	*Property*
1	Interior angles at e and e' are at most 45^0 each.
2	Vertex e (respectively, e') is the only vertex from which the entire left-wing (respectively, right-wing) can be illuminated by a 45^0-light.
3	(a) Edges $\overline{(a,b)}$, $\overline{(c,d)}$, and $\overline{(c',b')}$ are parallel to each other and their extensions meet segment $\overline{(i,j)}$ near its middle (b) Analogous properties hold for the edges $\overline{(a',b')}$, $\overline{(c',d')}$, and $\overline{(c,b)}$.
4	The mouth can be illuminated only from the vertex 45^0 light placed either at j or at j'.
5	The angles subtended by segment $\overline{(d,d')}$ at vertices j and j' are less than 45^0 each.

Lemma 1: *Three vertex 45^0-lights are necessary and sufficient to illuminate a beam-machine.*

A beam-machine can be illuminated by three vertex 45^0-lights by placing them either at (1) e, e', and j, or at (2) e, e', and j'; and we refer to these placements as **placement-j** and **placement-j'**, respectively.

Observation 1 (*Beam Formation*): Consider the illumination of a beam-machine by placement-j. In addition to the illumination of the interior of the beam-machine, it produces a thin beam of light in the exterior extending from the opening of the mouth and inclined to the right side (Figure 1a). Similarly, **placement-j'** produces a thin beam extending to the left. Illumination of the beam-machine by placement-j and placement-j' can be viewed as setting the beam-machine to "true" state and "false" state, respectively.

3.2 Beam Adjustment

It is desirable to be able to adjust the width and orientation of a beam without changing the visibility properties of the corresponding beam-machine. The width of the beam can be adjusted by narrowing or widening the opening in the mouth of the beam-machine, which can be done by adjusting the lengths of the edges $\overline{(a,b)}$ and $\overline{(a',b')}$. It is easily seen that the visibility properties listed in Table 1 do not change by adjusting the lengths of these edges. To rotate the beam to the right, we slightly shift vertex j to the left along the edge $\overline{(j',j)}$ and adjust the chain $< f, g, h, i, j >$ so that the left-wing is still visible from vertex e. The adjustment is shown by dashed chain in Figure 1b. Since the angle subtended by the edge $\overline{(d,d')}$ at vertex j is slightly less than 45^0 (property 5 in Table 1), the mouth of the beam-machine remains illuminated from the 45^0-light appropriately placed at the new position of j. To rotate the beam to the left we shift j above along the edge $\overline{(i,j)}$. Again the visibility properties listed in Table 1 are not affected by such a change.

3.3 Variable Generators

Corresponding to each variable x_i used in the collection of clauses W, we conatruct a variable generator V_i. A variable generator V_i contains beam-machine structures corresponding to each literal formed by using variable x_i. A variable generator is formed by adding beam-machines and spikes to a simpler structure called the **background of the variable generator (BVG)**. The BVG itself is constructed by adding suitable spikes to an even suitable simpler structure called the **skeleton**. Skeleton and BVG are shown in Figure 2 and the variable generator is shown in Figure 4a.

The Background of Variable Generators (BVG)

BVG is a twenty two sided polygonal chain $< v_1, v_2, ..., v_{23} >$ (Figure 2d) whose interior can be partitioned into five polygonal regions: (i) **left-wing** $< v_2, v_3, ..., v_7 >$, (ii) **right-wing** $< v_{22}, v_{21}, ..., v_{17} >$, (iii) **left-arm** $< v_7, v_8, ..., v_{12} >$, (iv) **right-arm** $< v_{12}, v_{13}, ...v_{17} >$, and (v) **body** $< v_1, v_2, v_7, v_{12}, v_{17}, v_{22}, v_{23} >$. BVG satisfies the visibility properties enumerated in Table 2. BVG can be systematically constructed starting from a U-shaped polygon: The U-shaped polygon is such that it can be partitioned into three isothetic rectangles rr, cc, and ll as shown in

Figure 2a. Rectangles ll and rr are of width one unit and height three units each. The rectangle cc is of width two units and height one unit. We choose the origin of the coordinate system at the mid-point of the base of rectangle cc. We first construct the polygonal chain $< v_1, x, v_3, v_4, v_7, v_{11}, v_{12}, v_{13}, v_{17}, v_{20}, v_{21}, y, v_{23} >$ that represents the **skeleton** of the BVG. Vertices $v_1, x, v_3, v_{11}, v_{12}, v_{13}, v_{21}, y$, and v_{23} of the skeleton have coordinates $(-0.5, -0.5), (-0.5, 0), (-1, 0), (-2, 3), (0, 1), (2, 3), (1, 0), (0.5, 0)$, and $(0.5, -0.5)$, respectively (Figure 2b). Vertex v_4 is identified on the horizontal line through point $(0, 0.5)$ such that the angle subtended at v_4 by line segment $\overline{(v_3, v_{12})}$ is less than 45^0. Vertex v_7 is the intersection of line segments $\overline{(v_4, v_{12})}$ and $\overline{(x, v_{11})}$. Vertices v_{20} and v_{17} are identified similarly. The point p_r where the extension of edge $\overline{(v_4, v_7)}$ meets the edge $\overline{(v_{13}, v_{17})}$ is called the **right hit-point**. Similarly, the **left hit-point** p_l is defined on the edge $\overline{(v_7, v_{11})}$. The sub-edges $\overline{(v_7, p_l)}$ and $\overline{(v_{17}, p_r)}$ are referred to as **spike-sites**. Let o be the mid-point of the edge $\overline{v_1, v_{23}}$. Consider the decomposition of the skeleton into three **arm-induced components**:

(i) $Q_1 = < v_3, v_4, v_7, v_{17}, v_{20}, v_{21} >$

(ii) $Q_2 = < o, v_7, v_{11}, v_{12}, y, v_{23} >$

(iii) $Q_3 = < o, v_1, x, v_{12}, v13, v_{17} >$.

The decomposition is illustrated in Figure 3a and each components are shown in Figure 3b.

Observation 2 (*Convexity Property of the Skeleton*): Arm induced components Q_1, Q_2, and Q_3 of the skeleton are convex. Hence, the skeleton can be illuminated by three vertex 45^0-lights.

BVG is constructed by attaching spikes to the skeleton as described in the following procedure (see Figure 2c) . The purpose of spikes is to force the placement of 45^0-lights on selected vertices.

Procedure 1: (*Construction of BVG from skeleton*)

Step1: We attach **hook-spikes** $< v_8, v_9, v_{10}, v_{11} >$ and $< v_{13}, v_{14}, v_{15}, v_{16} >$ on edges $\overline{(v_7, v_{11})}$ and $\overline{(v_{13}, v_{17})}$, respectively, to make interior angles at v_{11} and v_{13} at most 45^0 each. (A hook-spike is a non-convex quadrilateral and hence the only way to illuminate it by one 45^0-light is to place it at the vertex opposite to the reflex vertex.)

Step 2: We attach **dent-spike** $< x, v_2, v_3 >$ on the edge $\overline{(x, v_3)}$ so that vertices v_2, v_3, and v_{20} are collinear. Similarly, we attach **dent-spike** $< y, v_{22}, v_{21} >$ on the edge $\overline{(y, v_{21})}$. (The purpose of dent-spikes is to make (i) vertex v_3 not visible from vertices v_{21} and v_{22} and (ii) vertex v_{21} not visible from vertices v_2 and v_3.)

Step 3: We attach **ridge spike** $< v_4, v_5, v_6 >$ on the edge $\overline{(v_4, v_7)}$ such that (i) interior angle at v_4 becomes at most 45^0 and (ii) vertices v_5, v_6, and v_{20} are collinear. Similarly, ridge-spike $< v_{18}, v_{19}, v_{20} >$ is attached to the edge $\overline{(v_{17}, v_{20})}$.)

BVG shown in Figure 2d is formed when spikes attached to the skeleton (Figure 2c) are blended to form a sigle chain.

Table 2

(Visibility Properties of BVG)

Pr.#	*Property*
1	Interior angles at v_4, v_{11}, v_{13}, and v_{20} are at most 45^0 each.
2	Vertex v_{11} (respectively v_{13}) is the only vertex from which the entire left-arm (respectively, right-arm) can be illuminated by a 45^0-light.
3	(a) Both left-wing and right-wing can be illuminated by one 45^0-light placed at v_4 or v_{20}. (b) No other placement of a single 45^0-light can illuminate both wings.
4	(a) Vertices v_4, v_6, v_7 are collinear and the line through them meets the edge $\overline{(v_{16}, v_{17})}$ at hit-point p_r. (b) Vertices v_{17}, v_{18}, and v_{20} are collinear and the line through them meets the edge $\overline{(v_7, v_8)}$ at hit-point p_l.
5	(a) Vertex v_{19} is not visible from vertices v_5, v_6, and v_7. (b) Vertex v_5 is not visible from vertices v_{17}, v_{18}, and v_{19}.
6	(a) Vertex v_{21} is not visible from vertices v_2 and v_3. (b) Vertex v_3 is not visible from vertices v_{21} and v_{22}.

Lemma 2: *Three 45^0-lights are necessary and sufficient to illuminate BVG.*

Proof: The arms and the body of the BVG can be illuminated by placing 45^0-lights at vertices v_{11} and v_{13}. In any illumination of BVG by vertex 45^0-lights, the presence of spikes $< v_8, v_9, v_{10}, v_{11} >$ and $< v_{13}, v_{14}, v_{15}, v_{16} >$ force us to place two lights at vertices v_{11} and v_{13}. Hence two lights are necessary to illuminate the arms and the body of BVG, and at least one more 45^0-light is needed to illuminate the remaining parts: the wings of BVG. Both wings of BVG can be illuminated by placing one 45^0-light at v_4 or v_{20}. $\square$

Construction of Variable Generator V_i

As mentioned earlier, a variable generator is obtained by attaching beam-machines and spikes on the arms of BVG. (It may be recalled that variable x_i occurs k_{i_1} times as x_i and k_{i_2} times as $\overline{x_i}$.) We attach k_{i_1} beam-machines $d_1, d_2, ..., d_{k_{i_1}}$ on the lower side of the edge $\overline{(v_{11}, v_{12})}$ in the left-arm and k_{i_2} beam-machines $b_1, b_2, ..., b_{k_{i_2}}$ on the lower side of the edge $\overline{(v_{12}, v_{13})}$ in the right-arm (Figure

3a). Corresponding to k_{i_1} beam-machines we attach k_{i_1} **triangular spikes** $q_1, q_2, ..., q_{k_{i_1}}$ on the edge $\overline{(v_7, v_8)}$ such that all spikes lie within the spike-site $\overline{(v_7, p_l)}$. Similarly, we attach k_{i_2} triangular spikes $s_1, s_2, ..., s_{k_{i_2}}$ on the edge $\overline{(v_{16}, v_{17})}$ such that all spikes lie within the spike-site $\overline{(v_{17}, p_r)}$.

Lemma 3: *(a)* $3(k_{i_1} + k_{i_2} + 1)$ 45^0*-lights are necessary and sufficient to illuminate the interior of the variable generator* V_i*. (b) In addition to the illumination of the interior of* V_i*,* $3(k_{i_1} + k_{i_2} + 1)$ 45^0*-lights can generate* k_{i_1} *or* k_{i_2} *beams extending away from the mouth of* V_i*.*

Proof: No matter how the background of V_i is illuminated, each beam-machine attached to V_i needs three 45^0-lights to illuminate its interior. Hence by Lemma 1, $3(k_{i_1} + k_{i_2})$ 45^0-lights are necessary and sufficient to illuminate the interior of beam-machines attached to V_i. Since the light comming from the vertices of beam-machines can not completely illuminate any arm or wing of V_i, three 45^0-lights are needed to illuminate the background of V_i, no matter how the interior of beam-machines are illuminated. Thus, $3(k_{i_1} + k_{i_2} + 1)$ 45^0-lights are necessary and sufficient to illuminate the background and the beam-machines of V_i. The remaining parts of V_i (the triangular spikes attached to the arms) can be illuminated without using extra 45^0-lights as follows: Assume that the beam-machines are illuminated by using placement-j. Then the beams emanating from the mouth of the beam-machines are inclined to the right. The beams comming from the k_{i_2} beam-machines attached to the right-arm can be used to illuminate the k_{i_2} triangular spikes in the right-arm. Since the width and orientation of beams can be adjusted without compromising the visibility properties of beam-machines, we can focus the beams precisely to illuminate the corresponding triangular spikes. Also, since the triangular spikes attached to the left-arm lie below the hit-point p_l, they can be illuminated by the 45^0-light placed at v_{20}; and this light also illuminates both wings of V_i. The k_{i_1} beams comming from the beam-machines in the left-arm are free and escape away from the mouth of V_i as shown in Figure 4a. (This way of placing the minimum number of 45^0-lights to illuminate the interior of a variable generator is referred to as **left-placement**.)

On the other hand, consider the illumination of beam-machines by using placement-j'. In this case, the beams emanating from the beam-machines are inclined to the left. The k_{i_1} beams emanating from beam-machines in the left-arm can be adjusted to illuminate the corresponding triangular spikes $q_1, q_2, ...$ in the left-arm. The triangular spikes $s_1, s_2, ...$ and the wings of V_i are illuminated by one 45^0-light placed at v_4. The k_{i_2} beams emanating from the right-arm are free to escape to the exterior of V_i. (This way of placing the minimum number of 45^0-lights to illuminate the interior of a variable generator is referred to as **right-placement**.) $\square$

Formation of Shadow-region: When the background of a variable generator (say, V_i) is illuminated by the minimum number (three) of 45^0-lights, a shadow-region bounded by two illuminated region is formed in its exterior. There is a critical role of such shadow-regions in the final construction of polygon P. The formation of shadow-region is illustrated in Figure 4b. The 45^0-lights placed at vertices v_{11} and v_{13} generate two illumination regions in the exterior of V_i, one inclined to the left and the other to the right. The region bounded by these two illumination regions is the shadow-region (shaded in the figure) induced by the background of V_i.

3.4 Polygon Construction

By inspecting the clauses and literals in the given SAT instance we construct a simple polygon P. We start with a quadrilateral $< A, B, C, D >$ whose interior angles at B, C, and D are 90^0, 90^0, and 45^0, respectively. We attach two thin triangular spikes T_1 and T_2 on the side $\overline{(B, C)}$ such that D is the only vertex from which both of them are visible. For each variable x_i, we construct variable generator V_i containing $k_{i_1} + k_{i_2}$ beam-machines and attach them on the side $\overline{(A, B)}$. We also attach k triangular spikes on the side $\overline{(D, C)}$ corresponding to k clauses in the given SAT instance. The resulting structure is the required simple polygon P (Figure 5). Triangular spikes lying on the side $\overline{(D, C)}$ are the clause checkers which can be illuminated by beams comming from the variable generators. The size of the quadrilateral is made large enough (compared to the size of variable generators) so that all clause checkers lie in the intersection of shadow-regions induced by variable generators. The beam-machines are adjusted so that the beams escaping from the variable generators can be focussed on respective clause checkers to illuminate them. It is easily seen that this construction takes polynomial time.

Lemma 4: *All clauses in W are satisfiable if and only if P can be illuminated by m or fewer 45^0-lights, where m can be expressed in term of the number of literals and the number of clauses.*

Proof: (only if part) Assume that there exists a truth assignment T that satisfies all clauses in W. This means that at least one literal evaluates “true” in each clause. Let R be the set of literals formed by picking one literal from each clause such that each picked literal evaluates “true” under assignment T. We call a variable **selected** if it is used to form a literal in the set R. If a selected variable x_i appears uncomplemented in the set R then we illuminate the corresponding variable generator V_i by using the “left-placement” of 45^0-lights. (We recall the definition of “left-placement” and “right-placement” in the proof of Lemma 3.) Similarly, if a selected variable x_i appears complemented in the set R then we illuminate V_i by using “right-placement”. If a variable is not used to form any literal in R then we illuminate the corresponding variable generator by using either the left-placement or the right-placement (it does not matter which one we choose). Since each variable generator is illuminated by using either the “left-placement” or the “right-placement”, the total number of 45^0-lights used to illuminate all variable generators (r in total) is given by $g = \sum_{i=1}^{r} 3(k_{i_1} + k_{i_2} + 1)$ (Lemma 3). The only way to illuminate spikes T_1 and T_2 by a vertex 45^0-light is to place it at vertex D and hence we place it there. Thus, all interior region of the polygon, excepting the clause checkers, can be illuminated by $m = g + 1$ 45^0-lights. It may be noted that all clause checkers are in the intersection of shadow-region induced by the background of the variable generators; and hence, the only way to illuminate them without using any extra light source is to use the beams escaping fron the mouth of the variable generators. Since at least one literal evaluates "true" in each clause, there will be at least r beams escaping from the mouth of the variable generators. These beams are focussed on the clause checkers to illuminate them. Thus $m = g + 1$ 45^0-lights can illuminate P if all clauses in W are satisfiable.

(if part) Suppose that the polygon can be illuminated by $m = g+1$ 45^0-flood-lights. By Lemma 3, at least $g = \sum_{i=1}^{r} 3((k_{i_1} + k_{i_2}) + 1)$ 45^0-lights are needed to illuminate all variable generators. One 45^0-light is always needed at vertex D. Thus the clause checkers must have been illuminated by the beams escaping from the variable generators. If a clause checker corresponding to clause c_j is illuminated by the beam coming from the beam-machine on the left-wing of V_i then x_i must be uncomplemented in the clause c_j; hence we set x_i "true" to evaluate c_j "true". Similarly if the beam was coming from the right-wing then x_i must be uncomplemented in the clause c_j and hence we set x_i "false" to evaluate c_j "true". Since each clause checker is illuminated, all clauses can be evaluated "true" by the above assignment process. □

Lemma 4 implies that an instance of SAT can be reduced in polynomial time to an instance of FIP when $\alpha = 45^0$. Since SAT is known to be NP-complete FIP is NP-hard. This is stated in the following theorem

Theorem 1: *FIP is NP-hard when* $\alpha = 45^0$.

4. Illumination by α-lights

In this section we show how to generalize the reduction of Section 3 when the angular aperture α is fixed at any angle in the range $0 < \alpha \leq 360^0$. If α is greater than 45^0 then we can simply use the construction for 45^0-lights from the previous section. When α is smaller than 45^0 then we can modify the previous construction as follows. Some of the structures and terms used in the previous section will be prefixed by α to indicate the context of α-light. For example, terms beam-machine, skeleton, BVG, and variable generator V_i are referred to as α-beam-machines, α-skeleton, α-BVG, and α-V_i, respectively.

α-Beam-Machines: An α-beam-machine can be constructed in a straightforward way starting from a rectangle $< d, j, j', d' >$ (body of the *α-Beam-Machines*) whose diagonal makes angle $\alpha - \epsilon$

with its top side as shown in Figure 6, where ϵ is a small positive angle. We attach two right-angled triangles $< d, e, j >$ and $< d', e', j' >$ to the vertical sides of the rectangle; these triangles are such that the interior angles at e and e' are $\frac{\alpha}{2}$ each. We attach spikes $< e, f, g, h, i >$ and $< e', f', g', h', i' >$ on the edges $\overline{(e, j)}$ and $\overline{(e', j')}$ that satisfy properties 1 and 2 (for aperture angle α) in Table 1. Finally, we attach mouth satisfying properties 3, 4 in Table 1. In the resulting figure, we remove diagonals to obtain the required α-beam-machine. It is easy to observe that Lemma 1 is still valid: three α-lights are necessary and sufficient to illuminate an α-beam-machine.

Procedure 2 (*Construction of α-skeleton*): To construct α-skeleton, we start with the 45^0-skeleton (called simply skeleton in Section 3) and stretch its wings and arms by moving away vertices v_4, v_{11}, v_{13}, and v_{20} (Figure 7) so that the interior angles at v_{11} and v_{13} are reduced to α-δ each and that at v_4 and v_{20} are reduced to α-ϵ each, where ϵ and δ are positive angles each relatively smaller than α. While stretching, vertex v_{11} (respectively, v_{13}) is moved along the line passing through vertices v_{11} and v_{23} (respectively, v_{13} and v_1); and, vertices v_4 and v_{20} are moved along the horizontal line passing through them. The stretched skeleton so formed is the α-skeleton.

Lemma 5: *During the stretching of the skeleton as described in Procedure 2, the following invariants are maintained. (i) Locations of all vertices except for v_4, v_{11}, v_{13}, and v_{20} remain the same. (ii) The y-coordinates of v_4 and v_{20} remain equal to each other. (iii) The length of spike-sites $\overline{(v_7, p_l)}$ and $\overline{(v_{17}, p_l)}$ are non-zero. (iv) The arm induced components Q_1, Q_2, and Q_3 remain convex.*

Proof: Invariants (i) and (ii) follow immediately from the definition of the stretching. (iii) When v_4 and v_{20} are moved horizontally the hit-points p_l and p_r move toward v_7 and v_{17}, respectively. Since vertices v_7 and v_{17} are fixed and their y-coordinates are same, p_l and p_r coincide with v_7 and v_{17}, respectively, in the limit when the interior angles at v_4 and v_{20} become zero. Hence the lengths of spike-sites $\overline{(v_{17}, p_r)}$ and $\overline{(v_7, p_l)}$ remain non-zero as long as interior angles at v_4 and v_{20} are non-zero. (iv) The interior angles of the arm induced component Q_1 at vertices v_3, v_7, v_{17}, and v_{21} increase as stretching proceeds and become 180^0 in the limit when interior angles at v_4 and v_{20} become zero. Hence as long as interior angles at v_4 and v_{20} are non-zero all interior angles of the vertices of Q_1 are less than 180^0, implying that Q_1 remain convex. The convexity of Q_2 and Q_3 is argued similarly. □

Lemma 6: *α-BVG satisfying the corresponding visibility properties (restated by replacing 45^0 with α) listed in Table 2 can be constructed in polynomial time.*

Proof: We first construct α-skeleton with interior angles at v_4, v_{20}, v_{11}, and v_{13} equal to α-ϵ, α-ϵ, α-δ, and α-δ, respectively. Given a 45^0 skeleton, we can construct α-skeleton in linear time by stretching arms and wings as described in *Procedure 2.* Invariant iv (Lemma 5) guarantees that

α-skeleton can be illuminated by three α-lights placed either at v_4, v_{11}, and v_{13} or at v_{20}, v_{11}, and v_{13}.

We construct α-BVG by attaching hook-spikes, dent-spikes, and ridge-spikes to the α-skeleton so that the interior angles at v_4, v_{20}, v_{11}, and v_{13} become α each. This satisfies property 1, in Table 2. The convexity of Q_2 and Q_3 (Invariant iv, Lemma 5) and the presence of hook-spike imply that the placement of α-lights at v_{11} and v_{13} is necessary and sufficient to illuminate arms of α-BVG (property 2). Since v_7 and v_{17} have the same y-coordinates and are above the line through v_4 and v_{20}, we can attach ridge-spikes to satisfy properties 4 and 5. The convexity of Q_1 and the presence of ridge-spikes imply that one α-light placed either at v_4 or at v_{20} is necessary and sufficient to illuminate both wings (property 3). Since vertices v_3, x, y, and v_{20} do not move during the stretching, dent-spikes can be attached to satisfy property 6. Clearly, all these construction take polynomial time. □

Visibility properties listed in Table 2 and the convexity property stated in Observation 2 were used to prove Lemma 2 for aperture angle 45^0. Lemma 5 and Lemma 6 show that these properties are valid for any aperture angle in the range $0 < \alpha \leq 45^0$. Hence Lemma 2 is valid for any aperture angle in the range $0 < \alpha \leq 45^0$.

We construct variable generator α-V_i by attaching α-beam-machines and triangular-spikes on α-BVG in the same way as in Section 3. Invariant ii (Lemma 5) guarantees that the lengths of each spike-site is greater than zero when $\alpha > 0$. Hence triangular spikes can be attached on spike-sites $\overline{(v_7, p_l)}$ and $\overline{(v_{17}, p_l)}$ so that they can be illuminated either by the light from α-beam-machines or by the light from the tip of the wing. So the proof sketch of Lemma 3 for aperture angle 45^0 works for any aperture angle in the range $0 < \alpha \leq 45^0$.

The construction of the final polygon using α-V_i's is straightforward. The interior angle at vertex D is made equal to α. Observe that the top point H of the shadow region advances toward o (the mid-point of the edge $\overline{(v_1, v_{23})}$ as α decreases and coincides with o when α is zero. Hence clause checkers can be attached at the bottom of the polygon so that they lie on the intersection of the shadow regions induced by α-V_i's. This implies that Lemma 4 remains valid for any aperture angle in the range $0 < \alpha \leq 45^0$.

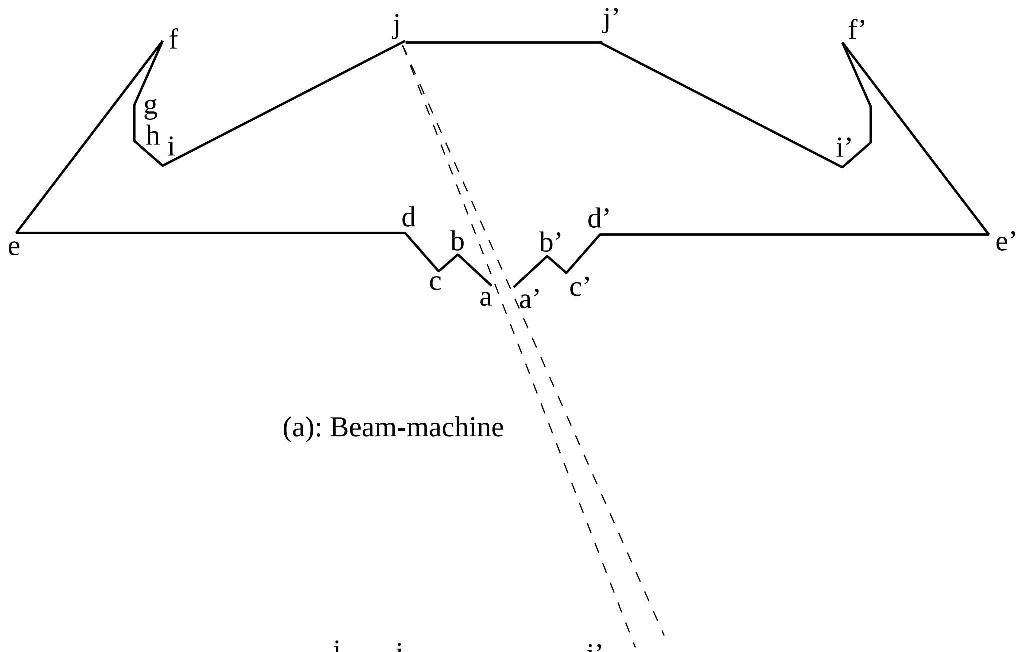
Theorem 2: *FIP is NP-hard.*

5. Discussion

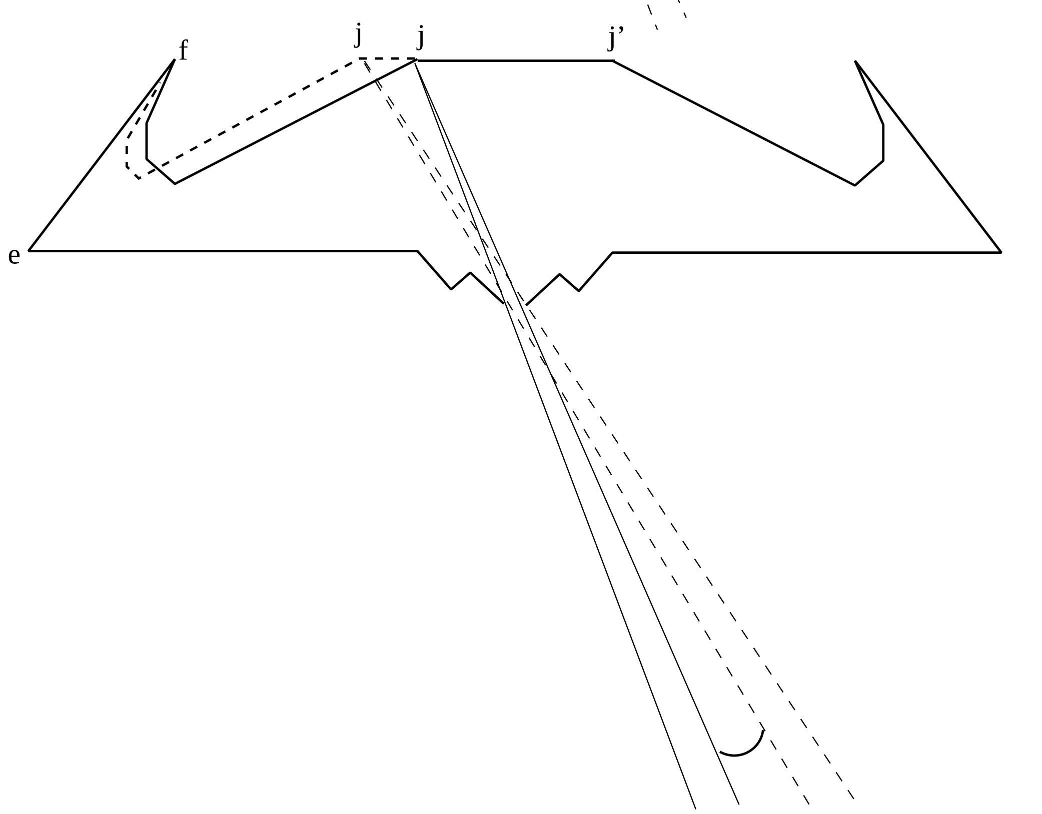
We have shown that the problem of illuminating a simple polygon with the minimum number of α-lights is NP-hard for any α in the range $0 < \alpha \leq 360^0$. Our construction can be modified to show that the problem remains NP-hard even if α-lights are required to flush with the edges of the polygon. Note that the polygon P constructed in our reduction is such that except for the α-lights placed in the beam-machines, all other α-lights are flush with the edges of the polygon. We can modify the beam-machine of Figure 1 to force α-lights to flush with the edges as follows. (The modified beam-machine is shown in Figure 8.) We replace the edge $\overline{(j,j')}$ by two edges $\overline{(j,o)}$ and $\overline{(o,j')}$, where o is the point of intersection of line segments $\overline{(j,d')}$ and $\overline{(j',d)}$. This modification does not effect the properties listed in Table 1. The construction of variable generators and the final polygon is similar to the construction described in Section 3. In the final construction all α-lights are forced to flush with the edges of the polygon implying that the problem is NP-hard. When the flushing condition is required we can check in polynomial time whether or not a given set of vertex α-lights can illuminate the polygon (there are at most two distinct orientations per vertex), implying that the problem is in the class NP. Hence, we have:

Corollary 1: *FIP is NP-complete if flood-lights are required to be flush with the edges of the polygon.*

The FIP problem becomes the well known art gallery problem when the visibility is allowed in all directions, i.e., when $\alpha = 360^0$. Although the art gallery problem for simple polygons is known to be NP-hard [1,2], it is still open for simple orthogonal polygons. We believe that an approach based on beam-machine construction might be helpful to settle this issue. When α-lights are required to flush with the edges, it may be possible to obtain a polynomial time algorithm to solve FIP.



(a): Beam-machine



(b): Adjusting the beam machine to rotate the beam

Figure 1: Illustrating beam generated from a beam machine

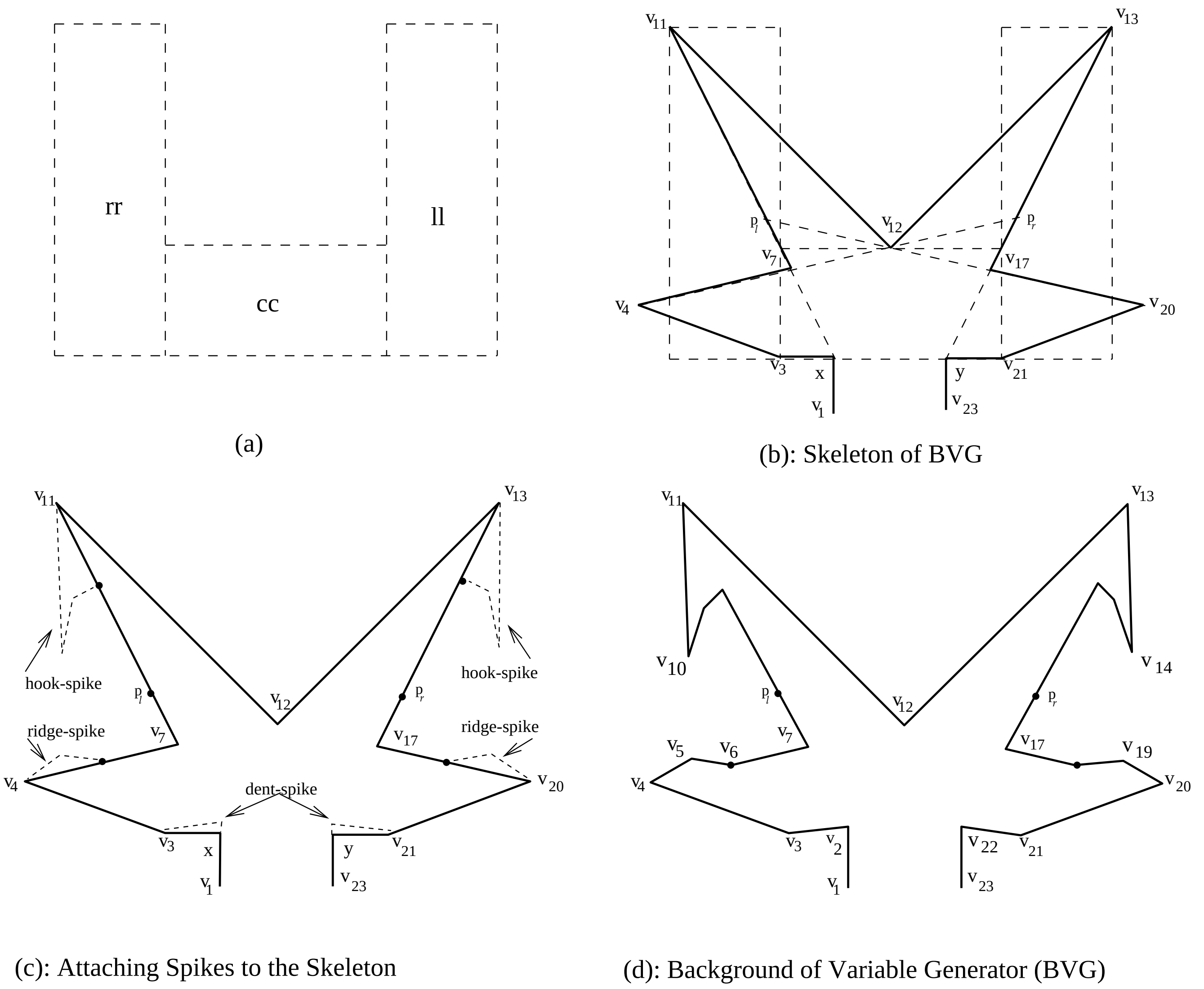


Figure 2: Illustrating the Construction of BVG

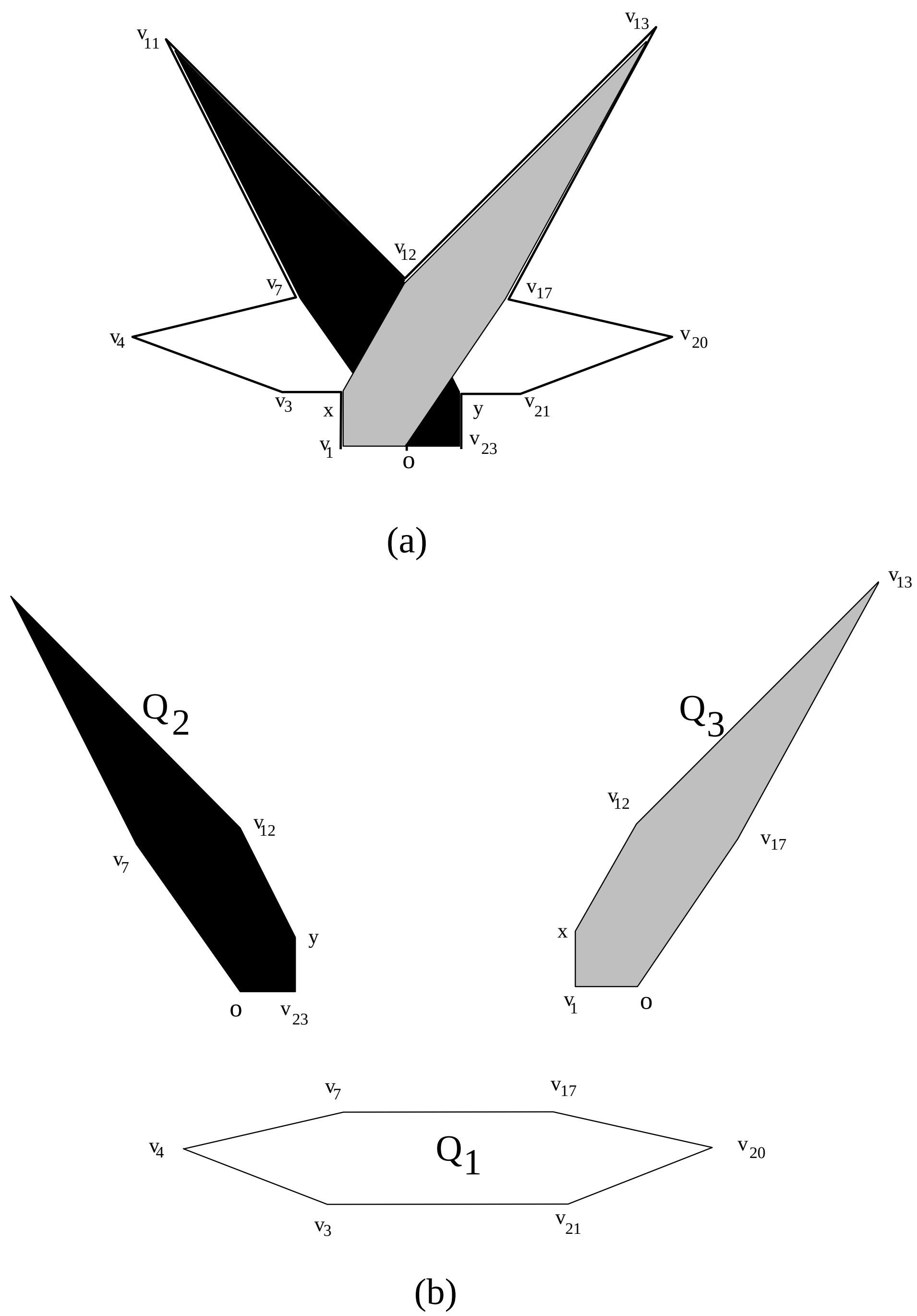
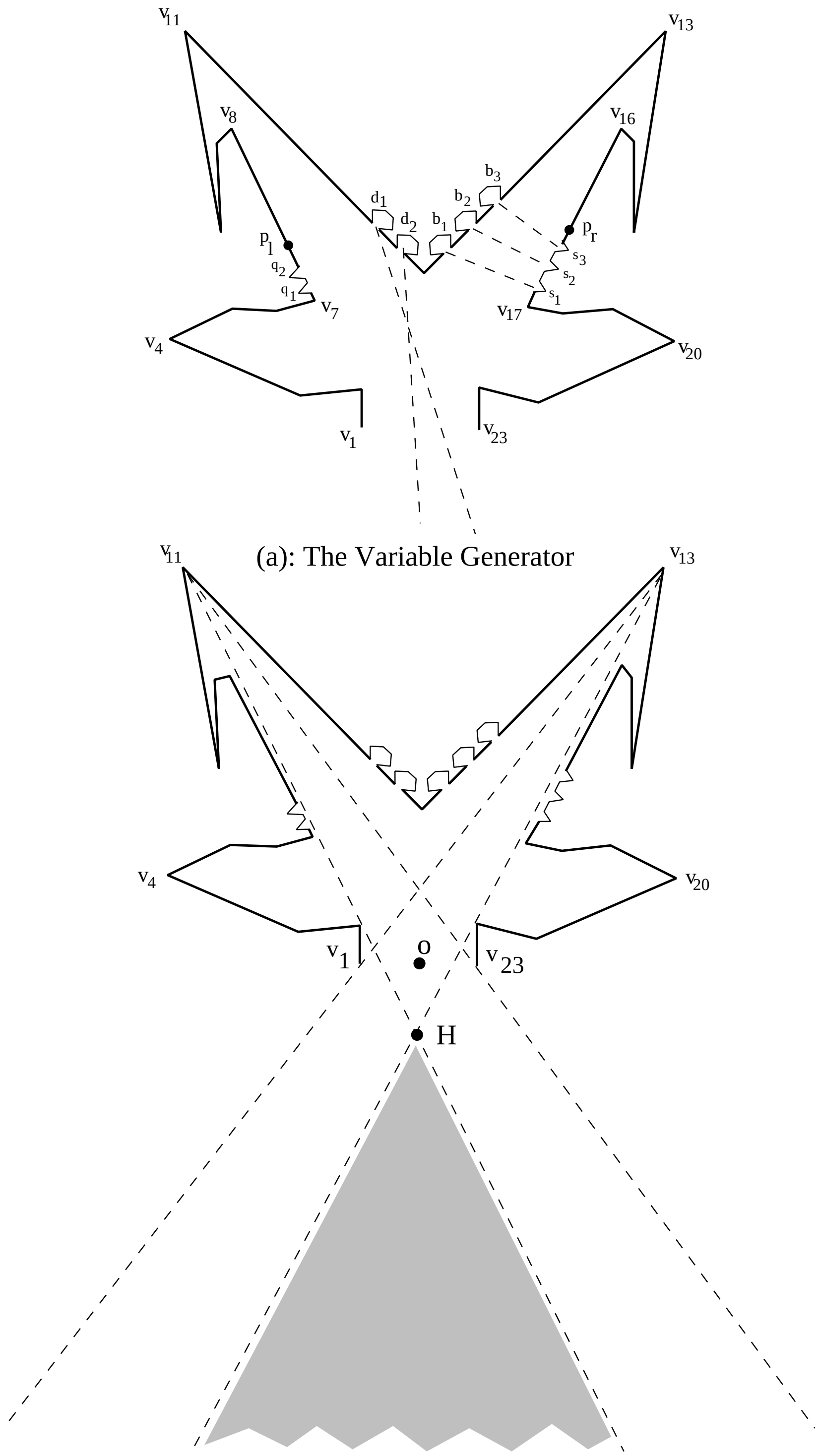


Figure 3: Illustrating Observation 2



(a): The Variable Generator

(b): The Shadow-Region Induced by the Variable Generator

Figure 4: Illustrating the Variable Generator

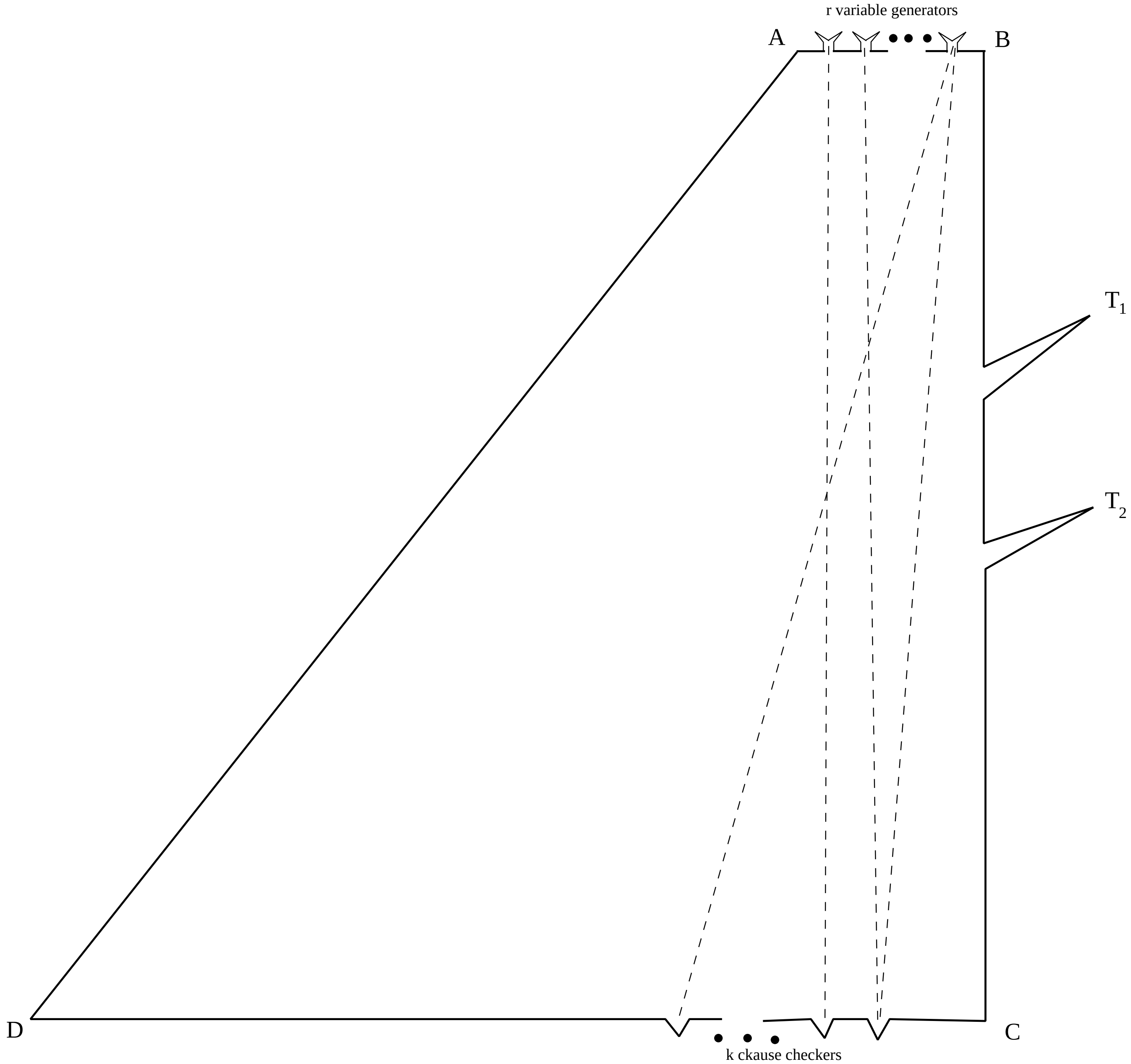


Figure 5: Illustrating the Construction of the Final Polygon

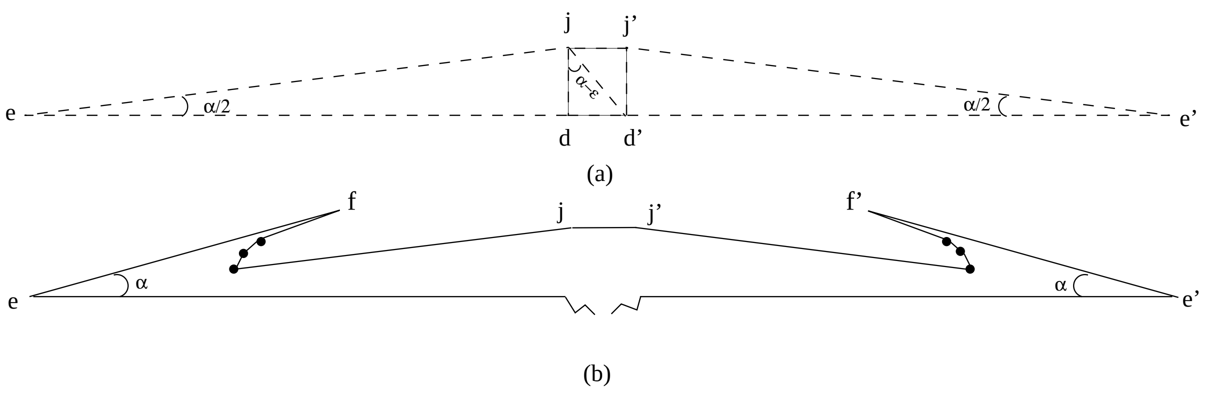


Figure 6: Construction of α-Beam-Machine

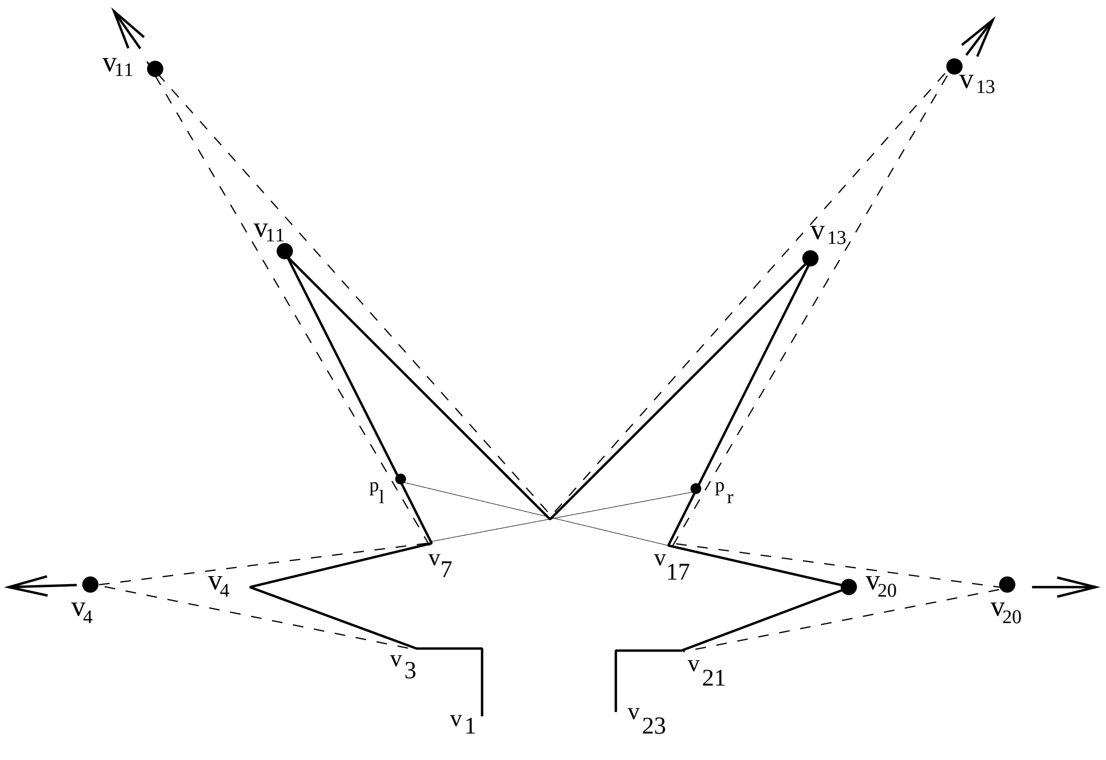


Figure 7: Illustrating the Stretching of the Skeleton

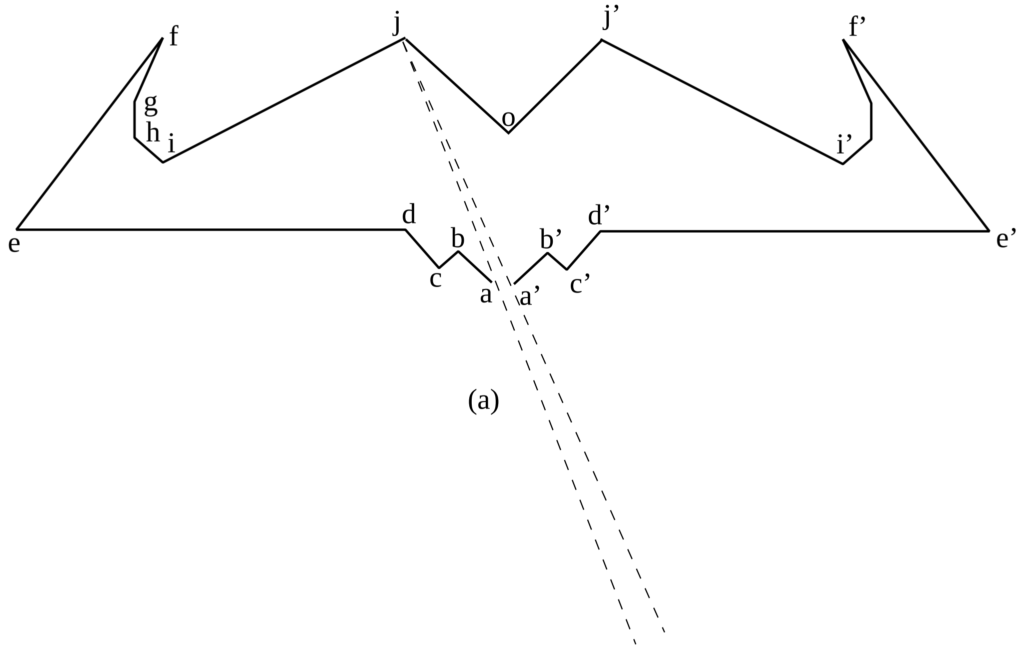


Figure 8: Modified Beam-Machine

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