Internal, External, and Mixed Visibility Edges of Polygons

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Abstract

In this paper we consider bounds on the number of edges connecting non-consecutive vertices of a polygon by dividing them into three types: *internal, external*, and *mixed* visibility edges. The main contribution of this paper is to show that for any polygon with n vertices, the sum of the number of internal and the number of external visibility edges is at least $\lceil \frac{3n-1}{2} \rceil - 4$. Tight bounds for the number of mixed visibility edges are also obtained. We explore several properties of uniquely triangulatable polygons which are useful in establishing bounds for the number of visibility edges. We also consider bounds for the number of visibility edges in terms of the number of convex vertices.

1 Introduction

Let P be a polygon with n vertices $v_1, \dots, v_n$. The *internal visibility graph* of P is the graph with vertex set $v_1, \dots, v_n$ in which two vertices are adjacent if the line segment connecting them does not intersect the exterior of P. The *external visibility graph* of P is defined in a similar way, except that the line segments that generate its edges are not allowed to intersect the interior of P. A visibility edge is called (*strictly) internal* (resp. *(strictly) external*) if it is not an edge of P and lies in the interior (resp. exterior) of P; see Figure 1(a).

Most of the visibility problems dealing with polygonal shapes are discussed in [4, 6, 8]. Problems dealing with the size of visibility graphs have been considered by several researchers in recent years [2, 3, 7]. Shen and Edelsbrunner [7] and Campbell and Higgins [2] have shown that the visibility graph of a set of n non-intersecting line segments contains at least $5n-4$ edges. Hurtado and Noy [3] have established tight bounds for the size of visibility graphs of simple polygons. Their result is based on exhibiting a relationship between the number of triangulations and the number of reflex vertices of a simple polygon.

In this paper we explore bounds for the sum of the numbers i and e of internal and external visibility edges of polygons. We also consider *mixed edges* which are the line segments connecting non-consecutive vertices of the polygon that intersect polygon edges. Determining the sum of the

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number of internal visibility edges and the number of external visibility edges, $i+e$, of a simple polygon is an open problem in the combinatorics of visibility and illumination [5]. We settle this problem by showing that for any simple polygon with n vertices $i+e \geq \lceil \frac{3n-1}{2} \rceil - 4$; we show how to construct polygons that realize the stated bound. This relation leads to bounds for the number of mixed edges m: $0 \leq m \leq \binom{n}{2} - \lceil \frac{5n-1}{2} \rceil + 4$. We also present a class of polygons that realize the minimum value of $i+e$ when the number of convex vertices is specified. For any uniquely triangulatable n-vertex polygon with c convex vertices we establish the relation $6-n \leq 2c-n \leq 2$ if n is even, and $6-n \leq 2c-n \leq 3$, if n is odd. Several additional properties of uniquely triangulatable polygons are presented.

The paper is organized as follows. In Section 2, we establish tight bounds for the number of visibility edges $(i+e)$ and the number of mixed edges (m). We then explore bounds for $i+e$ and m when the polygon has a specified number of convex vertices, c. It turns out that polygons that can be triangulated in only one way play an important role in establishing bounds for the number of visibility edges. We present several properties of uniquely triangulatable polygons in Section 3. We conclude in Section 4 by discussing bounds for the number of visibility edges for restricted classes of polygons (convex, spiral, monotone), and bounds for the number of mixed edges of given order.

2 Bounds for the total number of visibility and mixed edges

In this section we establish tight bounds for the total number of visibility edges $(i+e)$ and mixed edges (m) for a simple polygon. A vertex v of P will be called *internal* if it is in the *interior* of the convex hull $Conv(P)$ of P. An *external vertex* is a vertex of the convex hull of P. We now prove:

Lemma 1 *If a n-vertex simple polygon has l internal vertices, then it can be decomposed into exactly $l+1$ convex components $P_1, P_2, P_3, \ldots, P_{l+1}$ in such a way that if n_i is the number of vertices in P_i then $n_1+n_2+\ldots+n_{l+1} = n+3l$.*

Proof: Consider all the internal vertices of P in some arbitrary order. For each internal vertex v we perform the following construction: starting at v draw a *partitioning line segment* that bisects the internal angle at v and extend it until it hits the boundary of P or a previously drawn partitioning line segment. If a partitioning segment hits an existing vertex, rotate it slightly so that it ends at an interior point of an edge; see Figure 3. Observe that the endpoints of any partitioning segment appear as vertices in exactly two of the resulting sub-polygons (technically, a vertex that is created on a previously drawn partitioning segment will appear in three sub-polygons but it will join two co-linear edges in one of the sub-polygons and we disregard it with respect to that sub-polygon). Therefore each partitioning segment contributes 4 vertices to $n_1, n_2, ..., n_{l+1}$. Hence, $n_1+n_2+...+n_{l+1} = (n-l)+4l = n+3l$ ∎

A partition of the polygon as described in the proof of Lemma 1 is referred to as a *convex Steiner partition*. Each component P_i has two types of vertices: *original vertices* of P and Steiner vertices introduced by the partition. Let $P_i^{'}$ be the convex polygon formed by the set of original vertices of P_i and let w_i be the number of vertices of $P_i^{'}$. Note that if $w_i \geq 4$, then any internal edge of P_i is intersected by at least $w_i - 3$ internal visibility edges of $P_i^{'}$. This is stated in the following observation.

Observation 1 *If $P_i^{'}$ has w_i vertices , $w_i \geq 4$, then any internal visibility edge of $P_i^{'}$ is intersected by at least $w_i - 3$ internal visibility edges of $P_i^{'}$.*

The following lemma relating the number of external edges e and the number of internal vertices l can be easily established.

Lemma 2 *If a simple polygon P has l internal vertices then $e \geq l$.*

We now have:

Theorem 1 *For any simple polygon P with n vertices, $i + e \geq \lceil \frac{3n-1}{2} \rceil - 4$.*

Proof: Suppose that P has l internal vertices. We partition P into $l+1$ convex polygons $P_1, P_2, \ldots, P_{l+1}$ as in Lemma 1. If $w_i \geq 4$, we select an internal visibility edge s_i of $P_i^{'}$, $i = 1, 2 \ldots, l+1$. The set of these selected edges is referred to as the *skeleton set*. We consider an interior triangulation T of P such that T contains all the members of the skeleton set. By Observation 1, for each $w_i \geq 4$, an edge in the skeleton set is intersected by at least $w_i - 3$ internal edges. Hence the total number of internal edges i can be written as:

$$i \geq (n-3) + \sum_{i=1,\ldots,l+1} (w_i - 3)$$

We observe that each external vertex of P appears only in one component, and each internal vertex of P appears in exactly two components. Hence we can write

$$\sum_{i=1,\ldots,l+1} (w_i - 3) = ((n-l) + 2l) - 3(l+1) = n - 2l - 3$$

Combining the above two relations we get

$$i \geq 2n - 2l - 6$$

Combining Lemma 2 and the above relation we get

$$i + e \geq 2n - l - 6$$

Notice that $i \geq n - 3$ (any triangulation of a polygon has $n - 3$ edges), and hence we have:

$$i + e \geq (n-3) + l$$

Using these two relations to eliminate l we get $\quad i + e \geq \lceil \frac{3n-1}{2} \rceil - 4$ ∎

Figure 4 shows polygons realizing the bounds stated in Theorem 1. Since all the line segments connecting pairs of vertices of P (that are not edges of P) generate internal, external or mixed visibility edges, we have that $i+e+m=\binom{n}{2}-n$. Using this and Theorem 1 we now have

Corollary 1 $0 \leq m \leq \binom{n}{2} - \lceil \frac{5n-1}{2} \rceil + 4$.

2.1 Bounds on $i+e$ for polygons with c convex vertices

We now proceed to explore bounds for the total number of visibility edges $(i+e)$ when the number of convex vertices c is specified. A simple n-vertex polygon (with given c) is called *i-minimal* if it realizes the minimum number of internal edges and we denote the number of internal edges in such a polygon by $i(n,c)$. Similarly, we call a polygon *e-minimal* if it realizes the minimum number of external edges and we denote the number of external edges in such a polygon by $e(n,c)$. If a polygon realizes the minimum value of the sum $i+e$ then such a polygon is called *i-e-minimal*.

Theorem 2 *For a given c we can always construct a i-e-minimal polygon which is simultaneously i-minimal and e-minimal, i.e., if a polygon is i-e-minimal then* $i+e=i(n,c)+e(n,c)$.

To prove our result, we first obtain $i(n,c)$, and $e(n,c)$, and then we show the existence of polygons which attain both bounds simultaneously.

The following result by Hurtado and Noy [3] will be useful:

Lemma 3 *Any simple n-vertex polygon P with k reflex vertices contains $k+1$ convex sub-polygons, with disjoint interiors, $C_1, C_2, \ldots, C_{k+1}$ which cover all the vertices of P.*

Noy and Hurtado call $C_1, C_2, \ldots, C_{k+1}$ the *bags* of P. Although the convex bags of Lemma 3 cover all the vertices of P, their union is not necessarily the whole of P (Figure 5). We refer to partitions that satisfy Lemma 3 as *vertex-spanning convex partitions.*

The following lemma follows from Theorem 3 in the next section:

Lemma 4 *For c in the range* $3 \leq c \leq \frac{n+3}{2}$, $i(n,c)=n-3$.

Figure 6 shows two classes of uniquely triangulatable polygons with maximal c for odd n (Figure 6a) and for even n (Figure 6b). When the number of convex vertices c becomes greater than $\frac{n+3}{2}$, the quantity $i(n,c)$ increases monotonically with c and reaches the maximum value $\binom{n}{2}-n$, as depicted in Figure 7. The following two lemmas show that the growth is piecewise linear with progressively larger rates of growth and progressively shorter segments.

Lemma 5 *For odd n, if* $\frac{n+3}{2} < c+1 \leq \frac{2n+4}{3}$ *then* $i(n,c+1)=i(n,c)+2$. *For even n, if* $\frac{n+2}{2} < c+1 \leq \frac{2n+3}{3}$ *then* $i(n,c+1)=i(n,c)+2$.

Proof: Let n be odd (the other case is similar). The limits on the number of convex vertices in the statement of the lemma are equivalent to $(r+1)+2 < c+1 \leq 2(r+1)+2$, where r is the number of reflex vertices in the polygon. From Lemma 3 we have that $r+1$ diagonals can be used to create $r+1$ convex bags that contain all the vertices of the polygon (one or more regions of the polygon may be outside any of the convex bags). The total number of internal edges can be viewed as consisting of a *basic set* of $n-3$ non-crossing internal edges (realized by any triangulation of the polygon) and some additional internal edges that cross internal edges in the basic set (*crossing set*). Consider the number of internal edges within any one of the $r+1$ convex bags. Within a convex bag, both the total number of internal edges and the number of crossing internal edges are quadratic functions of the number of vertices in the boundary of the convex bag. Since the total number of vertices in all the bags is fixed for any decomposition with $r+1$ convex bags, the sum of the internal edges in all the bags (as well as the sum of the crossing internal edges) is minimized when the sizes of the various bags are as close to equal as possible. That is, to minimize the total number of crossing internal edges within the convex bags, one needs to distribute the vertices as evenly as possible among the $r+1$ convex bags. The total number of vertices to distribute is at least $n+r$; see Hurtado and Noy [3]. To minimize the total number of crossing internal edges (and hence the total number of internal edges) in the polygon, the best we can do is to have as few vertices as possible to distribute to the convex bags and to have no crossing internal edges in the remainder of the polygon (outside the convex bags). Both of these are achievable as shown in Figure 6a.

At the lower bound, we go from a uniquely triangulatable polygon with $c = r+3$ containing $i(n,c)$ internal edges to a polygon with $c = r+5$; this means that there are $n+r = 3r+5 = 3(r+1)+2$ vertices to distribute, resulting in r bags of size 3 and two bags of size 4, i.e., two crossing internal edges for a net growth of two in the number of internal edges (as in Figure 6c). Within the specified range, we go from an n-vertex polygon with r reflex vertices to an n-vertex polygon with $r-1$ reflex vertices; the number of vertices to distribute goes from $c+r+r$ to $(c+1)+(r-1)+(r-1)$ for a net reduction of one while the number of convex bags goes from $r+1$ to r. Using the pigeonhole principle, this is equivalent to going from trying to fit $3x+y$ pigeons into x pigeonholes of size 4 to trying to fit $3x+y-1$ pigeons into $x-1$ pigeonholes of size 4. Then, the number of "full" pigeonholes (i.e., crossing internal edges) goes from y to $y+2$ for a net change of 2 (two pigeons in the "lost" pigeonhole have to go elsewhere). At the upper bound, we can have $2n+4 \bmod 3$ equal to 0, 1 or 2 depending on the value of n. If $2n+4 \bmod 3$ is 0, then we have $c = 2r+4$ (for the polygon that realizes $i(n,c+1)$); there are $4(r+1)$ vertices to distribute resulting in $r+1$ bags of size 4 and $r+1$ crossing internal edges ($i = n+3+(r+1)$ with one crossing internal edge per convex bag). If $2n+4$ is not a multiple of 3, then there will be one or two leftover convex bags of size 3 at the upper bound (in Figure 6d, we have n=15, $2n+4 \bmod 3$ is 1 and we have one bag of size 3 left over). ∎

Lemma 5 can be generalized to the following:

Lemma 6 *If* $\frac{kn+k+2}{k+1} < c+1 \leq \frac{(k+1)n+(k+1)+2}{(k+1)+1}$ *then* $i(n,c+1) = i(n,c) + \frac{k(k+3)}{2}$*, for odd* n*; if* n *is even, the bounds become* $\frac{kn+k+1}{k+1} < c+1 \leq \frac{(k+1)n+(k+1)+1}{(k+1)+1}$*, where* $k = 1, 2, \ldots$*.*

Proof: The limits on c in the statement of the lemma are equivalent to $k(r+1)+2 < c+1 \leq (k+1)(r+1)+2$, where r is the number of reflex vertices in the polygon. Using the same argument as in Lemma 5, we have convex bags of size $k+2$ and $k+3$. Going from an n-vertex polygon with c convex vertices to an n-vertex polygon with $c+1$ convex vertices amounts to replacing $k+2$ convex bags of size $k+2$ with $k+1$ convex bags of size $k+3$. The net increase in the number of crossing internal edges is

$$(k+1)(\binom{k+3}{2} - (k+3) - k)) \quad - \quad (k+2)(\binom{k+2}{2} - (k+2) - (k-1)) = \frac{k(k+3)}{2}$$

∎

Note that the growth in $i(n,c)$ is constant within each of the ranges for c that are specified in Lemma 6. Lemma 6 does not specify the growth of $i(n,c)$ for unit increases in c that lie between successive ranges of values of c. For example, if $n = 101$, $i(n,c)$ increases by 2 units for each unit increase in c in the range $53 \leq c \leq 68$; it increases by 5 units per unit increase in c in the range $69 \leq c \leq 77$ and by 9 units for c in the range $78 \leq c \leq 82$. The growth of $i(n,c)$ in the unit intervals between the ranges specified in Lemma 6 takes on intermediate values that depend on the remainders at the upper bound for the previous range. In our example, $i(n,c)$ grows by 5 units as c goes from 68 to 69 (an intermediate value because $2n+4 \bmod 3$ has a remainder of 2); it increases by 9 units as c goes from 77 to 78 (there is a smooth transition between the ranges because $3n+5$ is a multiple of 4). As c gets closer to n, there will be jumps of increasing magnitude in the sizes of the convex bags that are used. The exact value of the minimum $i+e$ is $i(n,c) + n - c$ and it can be easily computed for any n, c, $c > \frac{n+3}{2}$.

Corollary 2 *If* $c > \frac{n+3}{2}$ *then* $i(n, c+k) \geq i(n,c) + k$ *for* $k > 0$*.*

We can construct e-minimal polygons similarly. If $c \geq \frac{n+3}{2}$ then an e-minimal polygon is obtained by placing all convex vertices on a circular arc so that all of them lie on the convex hull boundary. The polygon formed in this way contains exactly $n-c$ external edges. If $c < \frac{n+3}{2}$ then we have an oversupply of reflex vertices and hence the number of external edges must grow since the excess of reflex vertices will create convex caves of increasing size in the exterior of the polygon P. Looking at the cave(s) formed in the region(s) between the convex hull of P and P itself, we have a situation similar to that in Lemmas 5, 6 (but now on the outside of P). Again, to minimize the number of external edges, we need to distribute the reflex vertices evenly among the convex caves. At the upper bound $c = \lfloor \frac{n+3}{2} \rfloor$ for a uniquely triangulatable polygon, we have that $c = r + 3$ (if n is odd) or $c = r + 2$ (if n is even) and we have r convex caves of size 3 between P and its convex hull. We can decrease c by one (r increases by one) without having to increase the size of any of the convex caves by creating an extra cave. This adds only one external edge. However, as c decreases further, there are not enough convex vertices to keep the reflex vertices apart and we have to create convex caves of size larger than 3. To minimize e, we replace the reflex vertices with chains of reflex vertices (with the sizes evenly distributed) so that all of them lie on a circle arc (Figure 6e). We therefore have the following lemma.

Lemma 7 *$e(n, c)$ increases piecewise linearly as c decreases. The slopes of the linear segments are $k(k + 1)/2$ and there are inflection points at $c = \lceil \frac{n+k}{k+1} \rceil$ (for odd n) or $c = \lceil \frac{n}{k+1} \rceil$ (for even n) for $k = 2, 3, \ldots$.*

Now we are ready to prove Theorem 2.

Proof of Theorem 2: We can construct polygons that simultaneously minimize i and e as follows. If $c = \frac{n+3}{2}$ then we can construct a uniquely triangulatable polygon (Lemma 4) as shown in Figure 4 which is clearly i-minimal. This polygon has $n-c$ external edges and hence is e-minimal. If $c > \frac{n+3}{2}$ then we construct a polygon with $n - c$ external edges whose interior consists of $r + 1$ convex pockets (Figure 5(c-d)). Since it has $n - c$ external edges, it is e-minimal and by Lemma 6, such a polygon is also i-minimal. If $c < \frac{n+3}{2}$ then we start with a uniquely triangulatable polygon and distribute convex vertices and reflex vertices on circular arcs (Figure 6e) so that the external pockets have vertices as close to equal as possible. Since the polygon is uniquely triangulatable, it is i-minimal; furthermore, by Lemma 7, the polygon constructed in such a way is also e-minimal. ∎

3 Uniquely triangulatable polygons

In general, a simple polygon can be triangulated in more than one way. In fact, a convex polygon admits an exponential number of triangulations, and the number of ways in which convex polygons can be triangulated is related to the *Catalan Numbers* [3]. We call a polygon *modest* if it admits only one triangulation. Figure 8 is an example of a modest polygon. One of our main objectives in this section is to prove:

Theorem 3 *For any modest polygon P, the bounds for the total number of convex vertices are: $6 - n \leq 2c - n \leq 2$ if n is even, and $6 - n \leq 2c - n \leq 3$, if n is odd; modest polygons can be constructed for all values of c within the stated range.*

From this result, Lemma 4 follows easily. We also establish several additional properties of uniquely triangulatable polygons. Consider a modest polygon P with n vertices and let c be the number of convex vertices in it. Let $g(P)$ denote the difference between the number of convex vertices and the number of reflex vertices of P, i.e., $g(P) = c - (n - c) = 2c - n$. We refer to $g(P)$ as the *eccentricity* of P. Since any polygon must have at least three convex vertices, it follows that $g(P) \geq 6 - n$. A modest polygon with smallest eccentricity is shown in Figure 1b.

The next lemma follows immediately.

Lemma 8 *The sub-polygons formed by partitioning a modest polygon by a diagonal (internal edge) are both modest.*

We now prove:

Lemma 9 *At most one endpoint of any diagonal of a modest polygon can be convex.*

Proof: Assume to the contrary that there is a diagonal $\overline{a, b}$ connecting convex vertices a and b as shown in Figure 9. Rotate the diagonal $\overline{a, b}$ clockwise around vertex a with the other endpoint sliding along the edge incident on b until it hits a vertex p of the polygon. Similarly, let q be the first vertex encountered when the diagonal is rotated in the counterclockwise direction around b. Note that p and/or q could be adjacent to b. Now, observe that the chain $<a, q, b, p>$ is a convex quadrilateral inside the polygon. Hence $\overline{q, p}$ is a diagonal intersecting the diagonal $\overline{a, b}$. This implies that P is not a modest polygon—a contradiction. ■

The following lemma can be established similarly.

Lemma 10 *Let $d = \overline{a, b}$ be a diagonal of a modest polygon P, where both a and b are reflex. Then d must be tangent to at least one of its end vertices.*

It is easy to see that modest polygons with $n = 3, 4, 5, 6, 7, 8,$ and 9 and having the maximum value of eccentricity $g(P) = 2c - n$ exist; see Figure 11. An inspection of these polygons shows that for $3 \leq n \leq 9$, the maximum value of $g(P)$ is 3 when n is odd and 2 when n is even.

Consider the partitioning of a modest polygon P by a diagonal into components P' and P''. By Lemma 9, there can not be a diagonal connecting two convex vertices. Similarly, by Lemma 10, a diagonal connecting reflex vertices must be tangent to at least one of its end vertices. Thus the partitions of a modest polygon by a diagonal can be distinguished into four types as shown in Figure 12. Let n_1 and n_2 be the number of vertices in P' and P'', respectively. Similarly, let c_1 and c_2 be the number of convex vertices in P' and P'', respectively. Since a diagonal partitions each of the internal angles at its connecting vertices, we must have $n_1 + n_2 = n + 2$. The total number of convex vertices in P' and in P'' depends on the type of the diagonal used for partitioning P. An inspection of the partitioning diagonals shown in Figure 12 reveals the following. For type 1 and type 3 partitions, the number of convex vertices in P, P', and P'' are related by $c_1 + c_2 = c + 2$. Similarly, for type 2 and type 4 partitions the relation is $c_1 + c_2 = c + 3$. We summarize the above discussion in the following observation.

Observation 2

$$\begin{aligned} c_1 + c_2 &\geq c + 2 && (1) \\ n_1 + n_2 &= n + 2 && (2) \end{aligned}$$

Remark 1 *It can be shown that the above observation holds true even if the polygon is not uniquely triangulatable.*

We proceed now to prove Theorem 3.

Proof of Theorem 3: (By induction on the number of vertices n of P). It is easy to see that modest polygons with $n = 3, 4, 5, 6, 7, 8,$ and 9 and having the maximum value of eccentricity $g(P) = 2c - n$ exist; see Figure 11. An examination of these polygons shows that the theorem is true for n in the range $3 \leq n \leq 9$. Consider a modest polygon with the number of vertices $n > 9$. The relation $6 - n \leq 2c - n$ follows from the simple observation that any simple polygon must have at least three convex vertices. We establish the upper bound by distinguishing two cases.

Case 1 (n is odd): We partition P by a diagonal into two polygons P' and P'' with number of vertices n_1 and n_2 respectively. Since n is odd (and $n_1 + n_2 = n + 2$), n_1 and n_2 can not be both odd or both even. So assume without loss of generality that n_1 is odd and n_2 is even. By the induction hypothesis we have:

$$\begin{aligned} 2c_1 - n_1 &\leq 3 && (3) \\ 2c_2 - n_2 &\leq 2 && (4) \end{aligned}$$

Using Relations 1 and 2 (Observation 2) we get:

$$\begin{aligned} 2c - n &\leq 2(c_1 + c_2 - 2) - (n_1 + n_2 - 2) \\ &= (2c_1 - n_1) + (2c_2 - n_2) - 2 \\ &\leq 3 \qquad \text{(using relations 3 and 4)} && (5) \end{aligned}$$

Case 2 (n is even): We partition P by a diagonal into sub-polygons P' and P'' each with even number of vertices. The fact that we can always partition P into two parts, each with even number of vertices, can be established as follows. Consider an internal edge in P connecting vertices a, b. If this edge partitions P into parts with even number of vertices we are done. If not, rotate the edge about b so that its other endpoint slides along a polygonal edge (c, a) or (a, c). If the rotation can be completed (i.e., c is visible to a), then the internal edge connecting a and c partitions P into two parts with even number of vertices. If the rotation is blocked by a vertex d, we have that b is visible to d and the above argument can be repeated. Eventually, an internal edge that partitions P into even parts will be found.

Since P' and P'' each have even number of vertices we have:

$$2c_1 - n_1 \leq 2 \tag{6}$$

$$2c_2 - n_2 \leq 2 \tag{7}$$

Using Observation 2, we can write $2c - n$ as:

$$\begin{aligned} 2c - n &\leq 2(c_1 + c_2 - 2) - (n_1 + n_2 - 2) \\ &\leq (2c_1 - n_1) + (2c_2 - n_2) - 2 \\ &\leq 2 \qquad \text{(using relations 6 and 7)} \end{aligned}$$

∎

We end this section by proving an additional property of modest polygons. Consider extending the polygonal edges and internal edges of P incident on reflex vertices in the interior of P until the extensions meet the boundary. We call such meeting points the *hit-points* induced by reflex vertices. The following lemma characterizes convex chains of modest polygons in terms of hit-points induced by reflex vertices.

Lemma 11 *Let a and b be any consecutive convex vertices of a modest polygon P. Then the edge $\overline{a, b}$ must contain hit-points induced by some reflex vertex.*

Proof: Let a' and b' be the vertices adjacent on a and b respectively (Figure 10a). Imagine a clockwise rotation of the line segment whose one endpoint is fixed at a and the other endpoint slides along the edge $\overline{b, b'}$. Let t_r be the first vertex encountered by the rotating line. Similarly, let t_l be the first vertex encountered when line $\overline{b, a}$ moves counterclockwise with point b fixed and point a moving. If t_l and t_r are distinct then we have two intersecting diagonals; this implies that P is not modest—a contradiction. Now consider the situation when t_l and t_r degenerate into a single point t as shown in Figure 10b. Let h_l and h_r be the points where the extensions of $\overline{b, t}$ and $\overline{a, t}$ meet the edges $\overline{a, a'}$ and $\overline{b, b'}$, respectively. Now observe that the polygonal edges incident on vertex t are contained below the wedge $<h_l, t, h_r>$, which implies that the *hit-points* induced by t are on the edge $\overline{a, b}$. ∎

4 Concluding Remarks

In this paper, we presented tight bounds on the number of internal, external, and mixed visibility edges of polygons. We also established several properties of uniquely triangulatable polygons.

The bounds for the number of visibility edges and mixed edges can be easily extended to some well known families of polygons such as star-shaped, monotone and convex polygons. Note that the "saw-tooth" polygons of Figure 6a-b are monotone. Hence the lower bounds for i, $i+e$ and the upper bound on m are the same for monotone and general simple polygons. The upper bound for i is $\binom{n}{2}$ - n and is realized by any convex polygon (this also realizes lower bounds of zero for e, m). The upper bound for e is the same as for general polygons (the polygon of Figure 1b is monotone with respect to a horizontal axis). Since convex polygons are also star-shaped, the upper bound for i and lower bounds for e, m are the same as for general polygons.

The lower bounds on i, $i+e$ for star-shaped polygons are also the same as for general polygons and are realized by modifying the saw-tooth polygon of Figure 6a into a wedge with the teeth radiating out from a base point and then shifting the base point outside the kernel. The upper bound for e is again the same as for general polygons (the polygon of Figure 2 is star-shaped).

For spiral polygons, the polygon in Figure 1b realizes the maximum e while a polygon with a single reflex vertex that only blocks its immediate neighbors realizes the upper bound for $i = \binom{n}{2} - n - 1)$ and lower bounds on e, m (of 1 and 0 respectively). The lower bound on $i + e$ is $(n-3)+(n-4)$ and is realized by a spiral that wraps on itself so that both the interior and the one external cave enclosed by the hull are modest (Figure 13).

The mixed edges of P can be further distinguished according to the number of polygonal edges they intersect: A mixed edge that intersects with k edges of the polygon is referred to as a *type-k* mixed edge, $1 \leq k \leq n-4$ (since a mixed edge can not intersect any of the polygonal edges incident on its end-vertices). The polygons of Figure 14 demonstrate that the number of type-k mixed edges can be $O(n^2)$ for any fixed k. It would be interesting to find exact bounds for type-k mixed edges both overall and for polygons that are i-e-minimal.

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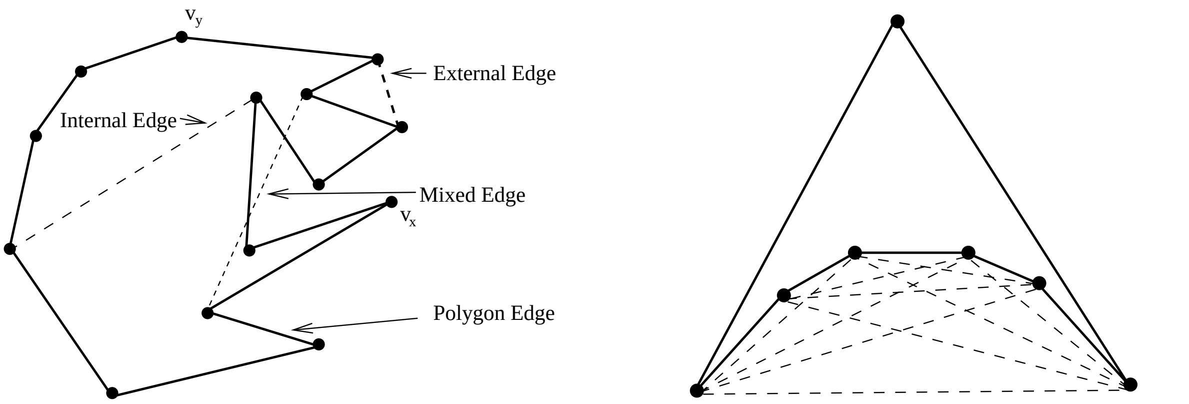

(a): Various Types of Edges

(b): A Polygon with Maximum Number of External Edges

Figure 1

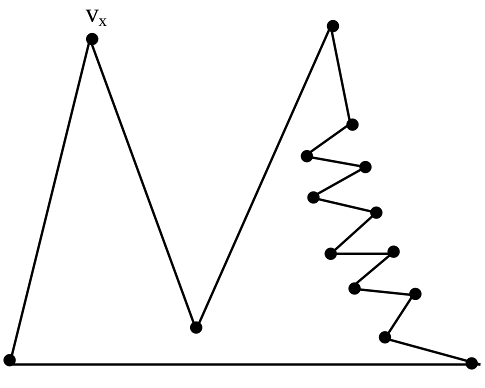

Figure 2: Achieving the Bound Stated in Corollary 1.1

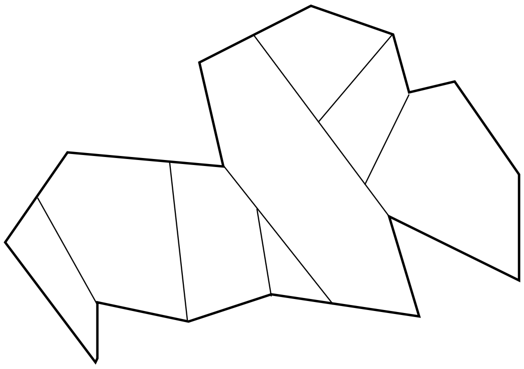

Figure 3: Proof of Lemma 3

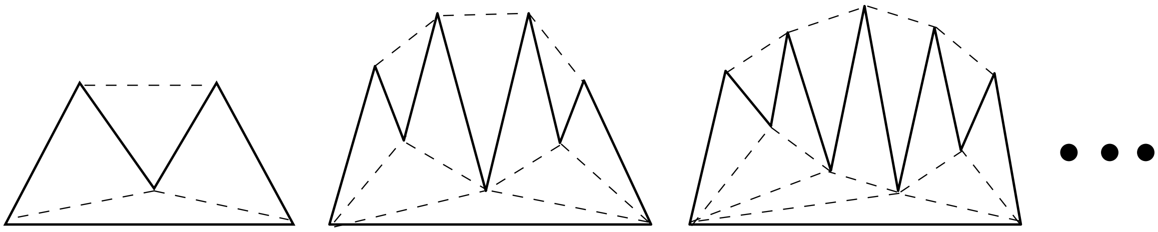

Figure 4: Polygons Achieving the Bounds Stated in Theorem 1

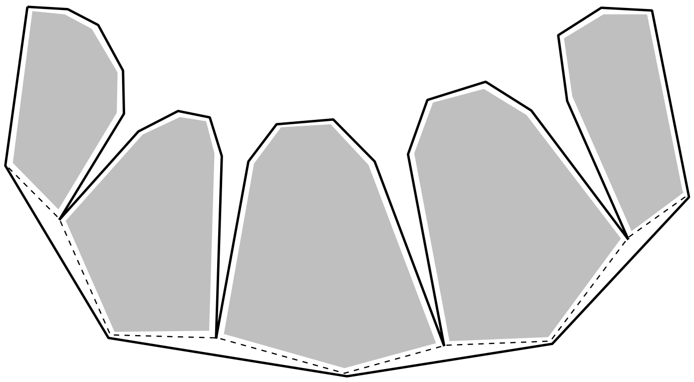

Figure 5: Illustrating Vertex Covering Convex Partitioning

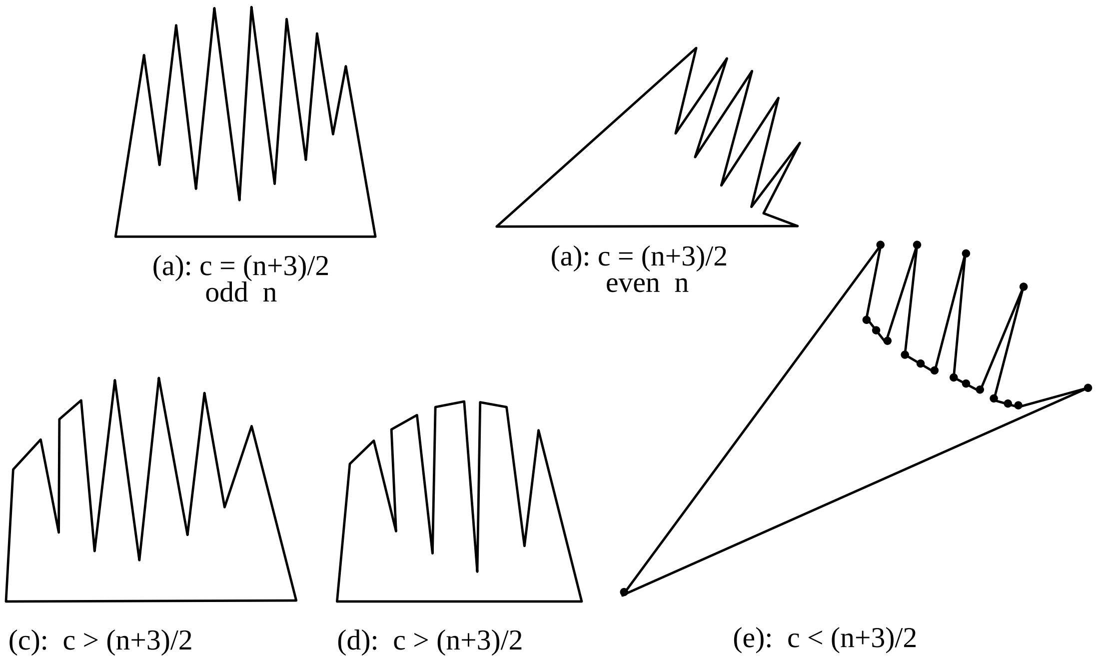

Figure 6: i-e-Minimal Polygons

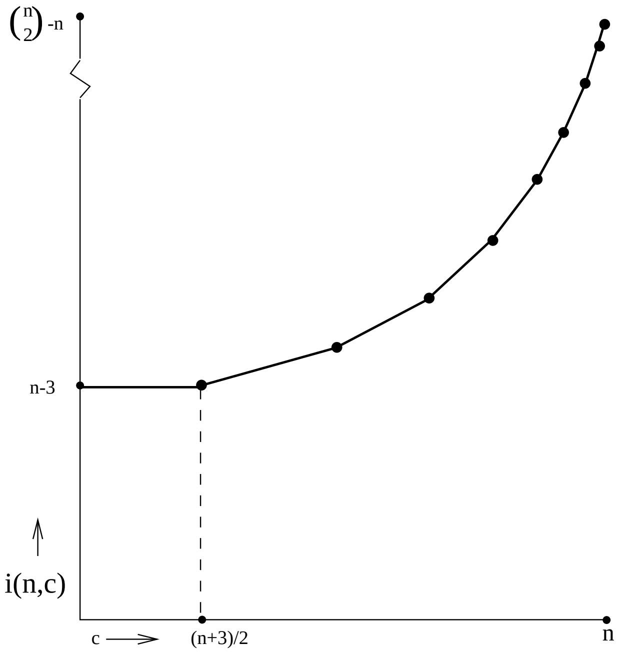

Figure 7: Variation of i(n,c) as a function of c

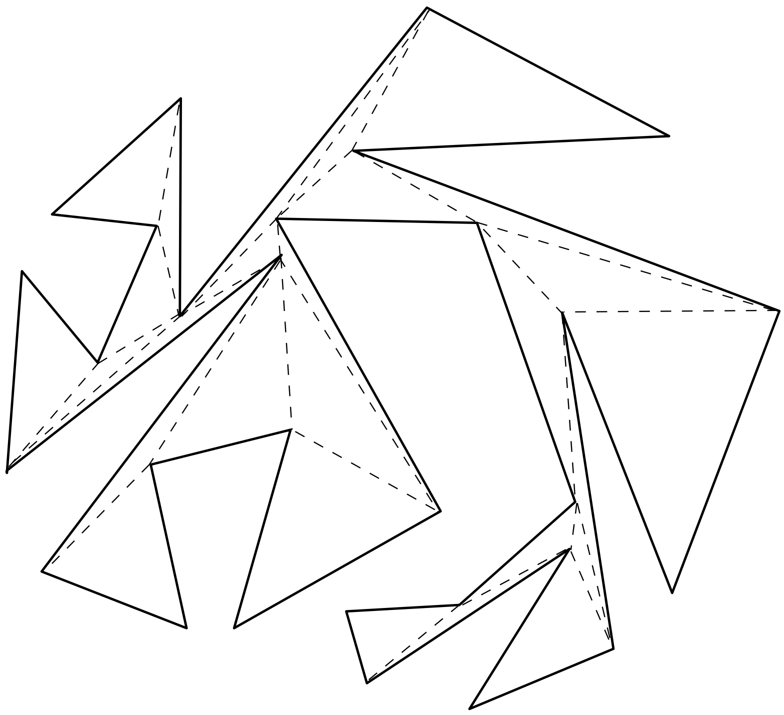

Figure 8: A Modest Polygon

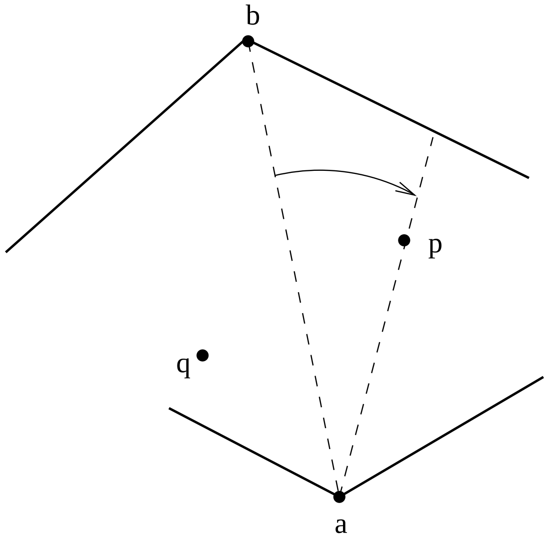

Figure 9: Proof of Lemma 9

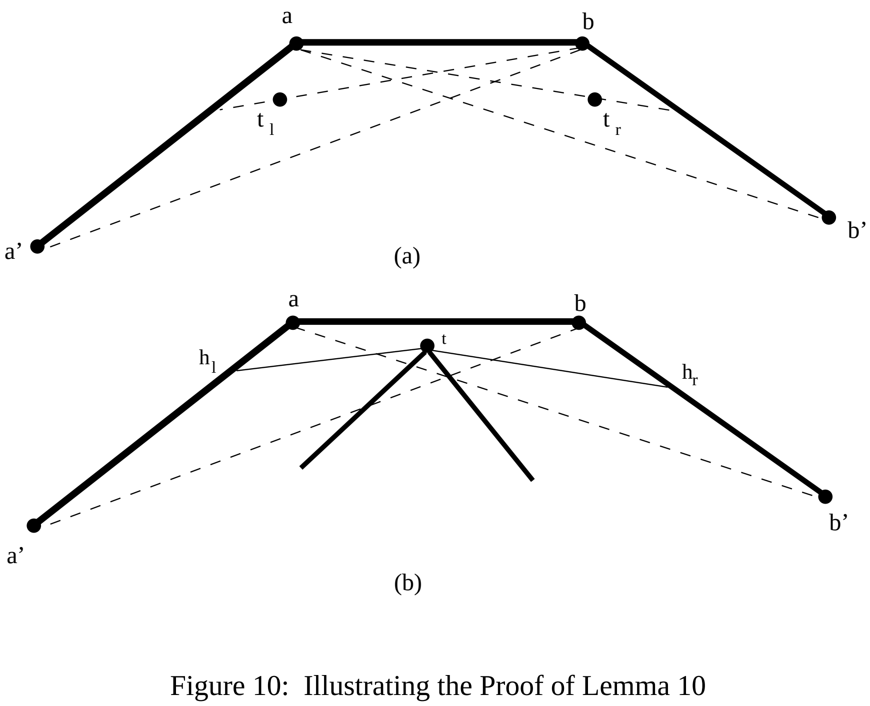

Figure 10: Illustrating the Proof of Lemma 10

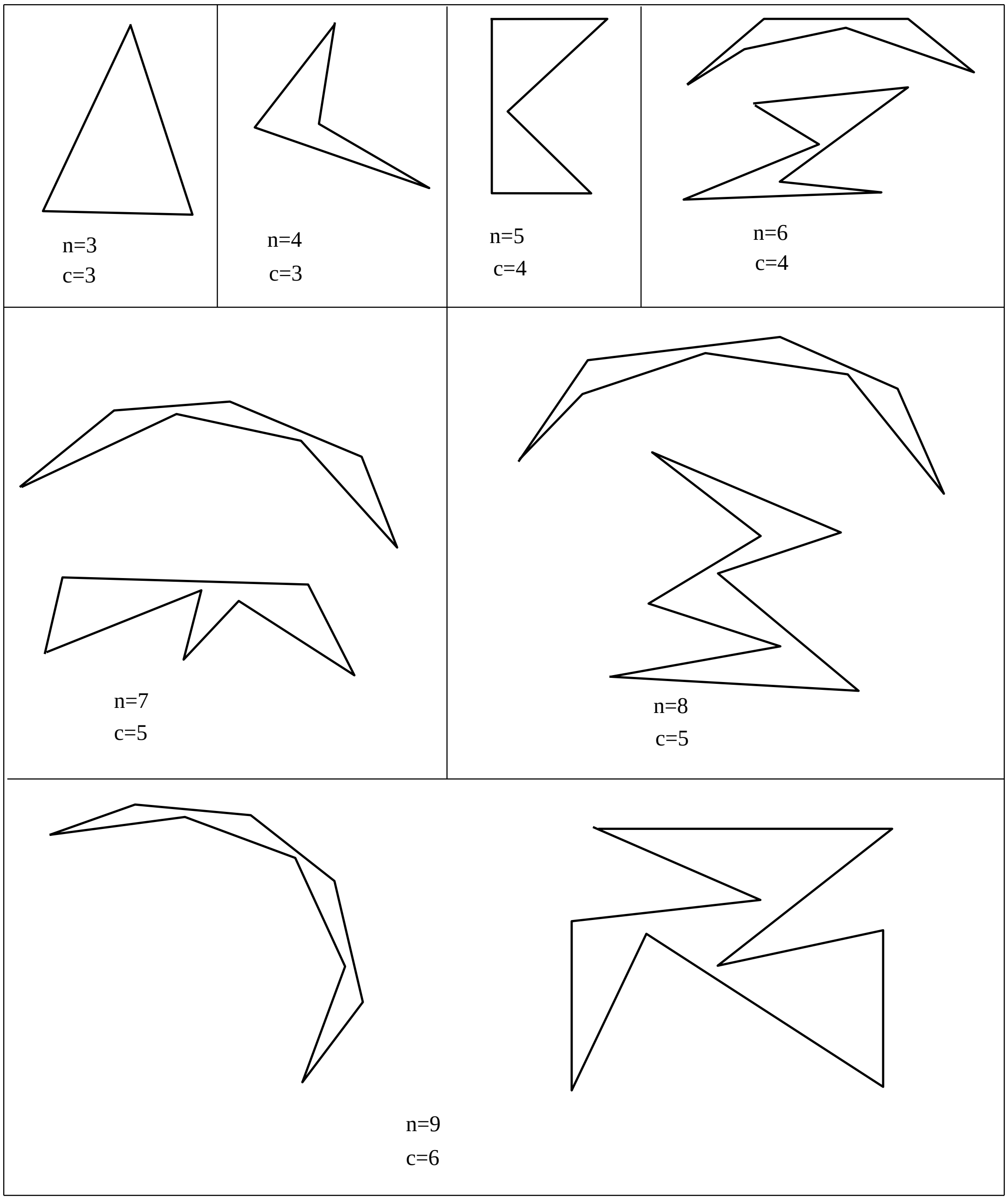

Figure 11: Examples of Modest Polygons with Maximum Number of Convex Vertices

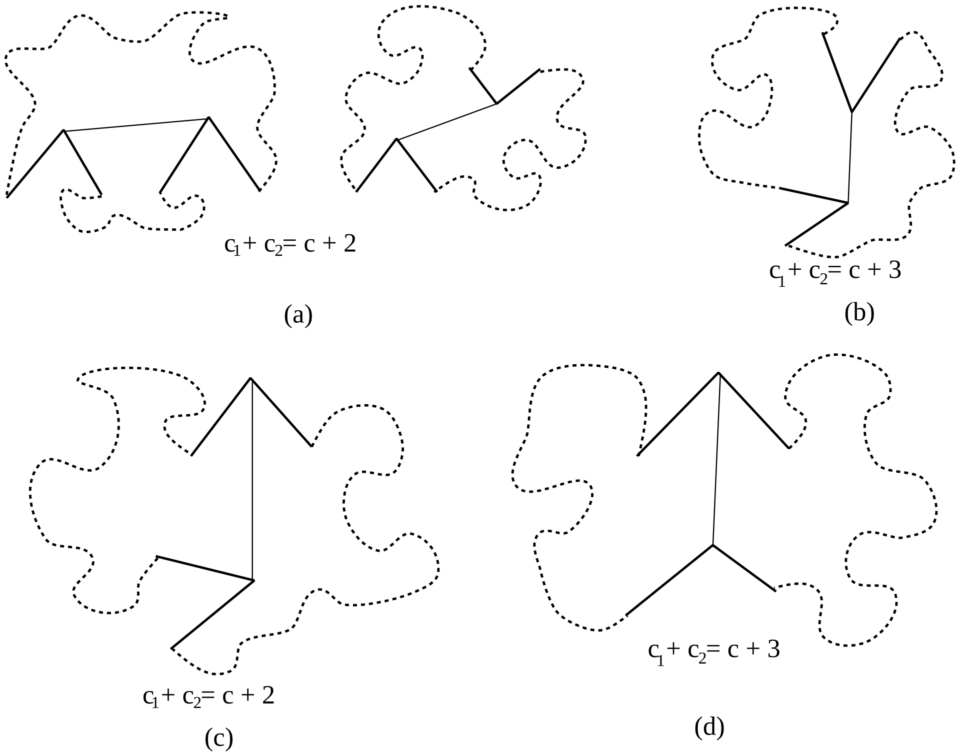

Figure 12: Illustrating Four Types of Partitions

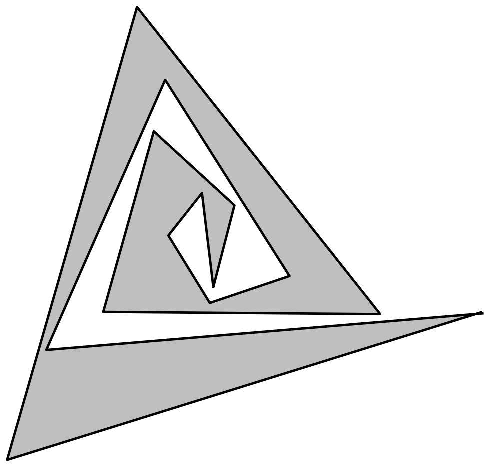

Figure 13

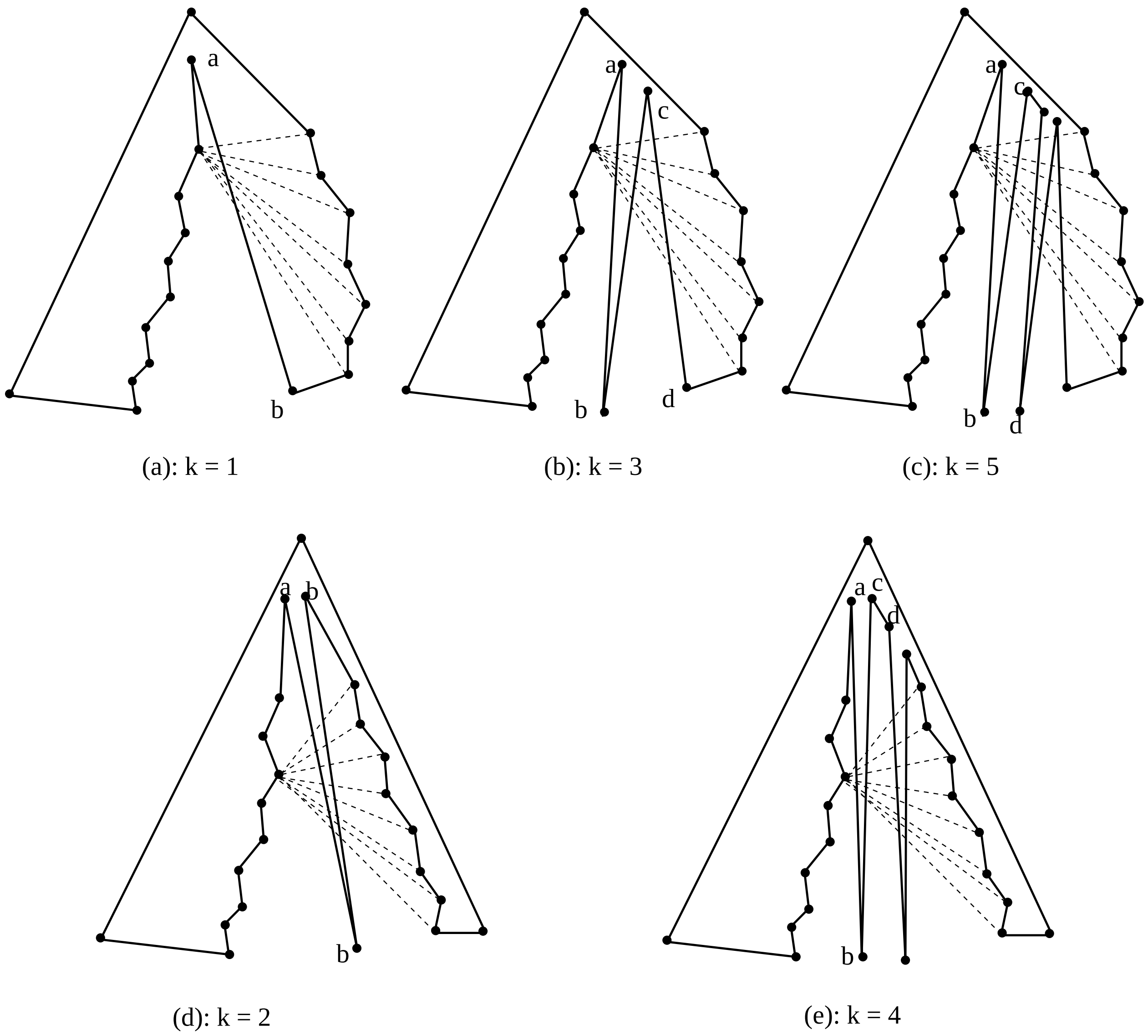

Figure 14: Polygons Exhibiting O(n^2) Mixed Edges of Order k = 1,2,3, ...