Homework #3
Due Su. 2/18

Note: The **Basic Problems with Answers** will be worth half as much as the other questions. You must show all your work to receive credit.

1. (OW 2.8) **Basic Problem with Answer**

   Solution

$$\begin{aligned} y(t) &= x(t) * h(t) = x(t) * [\delta(t+2) + 2\delta(t+1)] \\ &= x(t+2) + 2x(t+1) \end{aligned}$$

$$x(t+1) = \begin{cases} t+2 & 0 \le t+1 \le 1 \\ 2-(t+1) & 1 < t+1 \le 2 \\ 0 & \text{else} \end{cases} = \begin{cases} t+2 & -1 \le t \le 0 \\ 1-t & 0 < t \le 1 \\ 0 & \text{else} \end{cases}$$

$$x(t+2) = \begin{cases} t+3 & -2 \le t \le -1 \\ -t & -1 < t \le 0 \\ 0 & \text{else} \end{cases}$$

$$y(t) = \begin{cases} t+3 & -2 \le t \le -1 \\ 2(t+2)-t & -1 < t \le 0 \\ 2-2t & 0 < t \le 1 \\ 0 & \text{else} \end{cases} = \begin{cases} t+3 & -2 \le t \le -1 \\ t+4 & -1 < t \le 0 \\ 2-2t & 0 < t \le 1 \\ 0 & \text{else} \end{cases}$$

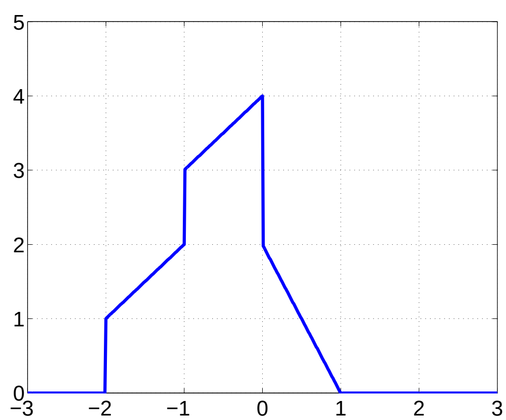

Figure 1: OW 2.8

2. (OW 2.9) **Basic Problem with Answer**

   Solution

$$\begin{aligned} h(t) &= e^{2t}u(-t+4) + e^{-2t}u(t-5) \\ h(t-\tau) &= e^{2(t-\tau)}u(-(t-\tau)+4) + e^{-2(t-\tau)}u(t-\tau-5) \\ &= e^{2(t-\tau)}\underbrace{u(\tau-t+4)}_{=0 \text{ for } \tau<t-4} + e^{-2(t-\tau)}\underbrace{u(-\tau+t-5)}_{=0 \text{ for } \tau>t-5} \end{aligned}$$

$$\Rightarrow A = t-5 \qquad B = t-4$$

3. (OW 2.10) **Basic Problem with Answer**

   Solution

   (a)

$$\begin{aligned}
&t < 0 && y(t) = 0\\
&0 \leq t < \alpha && y(t) = \int_0^t d\tau = t\\
&\alpha \leq t < 1 && y(t) = \int_{t-\alpha}^t d\tau = t - (t-\alpha) = \alpha\\
&1 \leq t \leq 1+\alpha && y(t) = \int_{t-\alpha}^1 d\tau = 1 - (t-\alpha) = 1+\alpha-t\\
&t > 1+\alpha && y(t) = 0
\end{aligned}$$

$$y(t) = \begin{cases} t & 0 \leq t < \alpha \\ \alpha & 1 \leq t < 1 \\ -t + (1+\alpha) & 1 \leq t \leq 1+\alpha \\ 0 & \text{else} \end{cases}$$

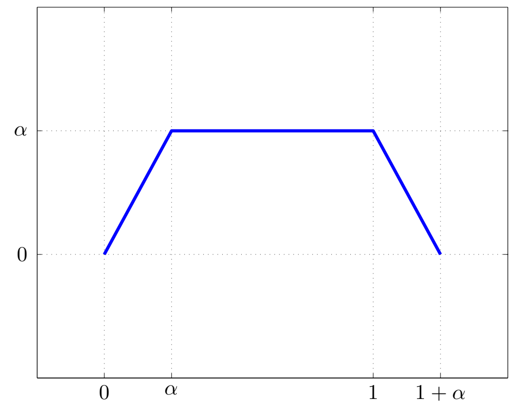

Figure 2: OW 2.10

   (b) $y(t)$ has 4 discontinuities at $t = 0, \alpha, 1,$ and $1+\alpha$ unless it is a triangle $\Rightarrow \alpha = 1$.

4. (OW 2.11) **Basic Problem with Answer**

   Solution

   (a)

$$\begin{aligned}
&t < 3 && y(t) = 0\\
&3 \leq t < 5 && y(t) = \int_0^{t-3} e^{-3\tau} d\tau = -\frac{1}{3}\left[e^{-3\tau}\right]_0^{t-3} = \frac{1}{3}\left[1 - e^{-3(t-3)}\right]\\
&t \geq 5 && y(t) = \int_{t-5}^{t-3} e^{-3\tau} d\tau = -\frac{1}{3}\left[e^{-3\tau}\right]_{t-5}^{t-3} = \frac{1}{3}\left[e^{-3(t-5)} - e^{-3(t-3)}\right]
\end{aligned}$$

$$y(t) = \begin{cases} \frac{1}{3}\left[1 - e^{-3(t-3)}\right] & 3 \le t < 5 \\ \frac{1}{3}\left[e^{-3(t-5)} - e^{-3(t-3)}\right] & t \ge 5 \\ 0 & \text{else} \end{cases}$$

(b)

$$d(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[u(t-3) - u(t-5)\right] = \delta(t-3) - \delta(t-5)$$
$$g(t) = d(t) * h(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)$$

(c)

$$\frac{d}{dt}y(t) = \begin{cases} e^{-3(t-3)} & 3 \le t < 5 \\ e^{-3(t-5)} - e^{-3(t-3)} & t \ge 5 \\ 0 & \text{else} \end{cases}$$
$$= g(t)$$

5. (OW 2.22)

Solution

(a)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$= \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau$$
$$= e^{-\beta t}\int_0^t e^{(-\alpha+\beta)\tau}d\tau \qquad t \ge 0$$

for $\beta = \alpha$

$$y(t) = e^{-\beta t}\int_0^t e^0 d\tau = e^{-\beta t}\left[\tau\right]_0^t = te^{-\beta t}$$
$$= te^{-\beta t}u(t)$$

for $\beta \neq \alpha$

$$y(t) = e^{-\beta t}\int_0^t e^{(-\alpha+\beta)\tau}d\tau = e^{-\beta t}\frac{1}{-\alpha+\beta}\left[e^{(-\alpha+\beta)\tau}\right]_0^t = \frac{e^{-\beta t}}{-\alpha+\beta}[e^{(-\alpha+\beta)t} - 1]$$
$$= \frac{e^{-\beta t}}{-\alpha+\beta}[e^{(-\alpha+\beta)t} - 1]u(t)$$

(b)

$$t > 6 \qquad y(t) = 0$$
$$3 \le t \le 6 \qquad y(t) = \int_{t-1}^{5}(-1)e^{2(t-\tau)}d\tau = \frac{1}{2}\left[e^{2(t-\tau)}\right]_{t-1}^{5} = \frac{1}{2}\left[e^{2(t-5)} - e^2\right]$$
$$1 \le t \le 3 \qquad y(t) = \int_{t-1}^{2} e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau = -\frac{1}{2}\left[e^{2(t-\tau)}\right]_{t-1}^{2} + \frac{1}{2}\left[e^{2(t-\tau)}\right]_2^5$$
$$= \frac{1}{2}\left[e^2 - 2e^{2(t-2)} + e^{2(t-5)}\right]$$
$$t < 1 \qquad y(t) = \int_0^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau = \frac{1}{2}\left[e^2 t - 2e^{2(t-2)} + e^{2(t-5)}\right]$$

$$y(t) = \begin{cases} \frac{1}{2}\left[e^2 t - 2e^{2(t-2)} + e^{2(t-5)}\right] & t < 1 \\ \frac{1}{2}\left[e^2 - 2e^{2(t-2)} + e^{2(t-5)}\right] & 1 \le t \le 3 \\ \frac{1}{2}\left[e^{2(t-5)} - e^2\right] & 3 \le t \le 6 \\ 0 & \text{else} \end{cases}$$

(c)

$$\begin{aligned} t < 1 \quad & y(t) = 0 \\ 1 \le t \le 3 \quad & y(t) = \int_0^{t-1} (2)\sin \pi\tau d\tau = -\frac{2}{\pi}\left[\cos \pi\tau\right]_0^{t-1} = \frac{2}{\pi}[1 - \cos \pi(t-1)] \\ 3 \le t \le 5 \quad & y(t) = \int_{t-3}^{2} (2)\sin \pi\tau d\tau = \frac{2}{\pi}\left[\cos \pi(t-3) - \cos \pi 2\right] = \frac{2}{\pi}\left[\cos \pi(t-3) - 1\right] \\ t > 6 \quad & y(t) = 0 \end{aligned}$$

$$y(t) = \begin{cases} \frac{2}{\pi}[1 - \cos \pi(t-1)] & 1 \le t \le 3 \\ \frac{2}{\pi}\left[\cos \pi(t-3) - 1\right] & 3 \le t \le 5 \\ 0 & \text{else} \end{cases}$$

(d)

Let

$$h(t) = h_1(t) - \frac{1}{3}\delta(t-2)$$

with

$$h_1(t) = \begin{cases} 4/3 & 0 \le t \le 1 \\ 0 & else. \end{cases}$$

$$y(t) = x(t) * h(t) = x(t) * [h_1(t) - \frac{1}{3}\delta(t-2) = \underbrace{x(t) * h_1(t)}_{g(t)} - \frac{1}{3}x(t-2)$$

$$\begin{aligned} g(t) &= \int_{t-1}^{t} \left(\frac{4}{3}\right) a\tau + b d\tau = \frac{4}{3}\left[(1/2)a\tau^2 + b\tau\right]_{t-1}^{t} \\ &= \frac{4}{3}[(1/2)at^2 + bt - (1/2)a(t^2 - 2t + 1) - b(t-1)] \\ &= \frac{4}{3}[at + b - \frac{1}{2}a] \\ y(t) &= g(t) - \frac{1}{3}x(t-2) = \frac{4}{3}[at + b - \frac{1}{2}a] - \frac{1}{3}x(t-2) \\ &= \frac{4}{3}[at + b - \frac{1}{2}a] - \frac{1}{3}[a(t-2) + b] = at + b \\ &= x(t) \end{aligned}$$

(e) Since $x(t)$ is periodic, $y(t)$ is periodic with $T = 2$ and the convolution only needs to be computed over a single period.

$$\begin{aligned} -\frac{1}{2} \le t \le \frac{1}{2} \qquad & y(t) = \int_{t-1}^{-1/2} (t - \tau - 1)d\tau + \int_{-(1/2)}^{t} (1 - t + \tau)d\tau = \frac{1}{4} + t - t^2 \\ \frac{1}{2} \le t \le \frac{3}{2} \qquad & y(t) = \int_{t-1}^{1/2} (1 - t + \tau)d\tau + \int_{1/2}^{t} (t - 1 - \tau)d\tau = t^2 - 3t + \frac{7}{4} \end{aligned}$$

$$y(t) = \begin{cases} \frac{1}{4} + t - t^2 & -\frac{1}{2} \le t \le \frac{1}{2} \\ t^2 - 3t + \frac{7}{4} & \frac{1}{2} \le t \le \frac{3}{2} \end{cases}$$

6. (OW 2.23)

Solution

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} \delta(t - kT) \\ y(t) = x(t) * h(t) &= \sum_{k=-\infty}^{\infty} \delta(t - kT) * h(t) \\ &= \sum_{k=-\infty}^{\infty} h(t - kT) \end{aligned}$$

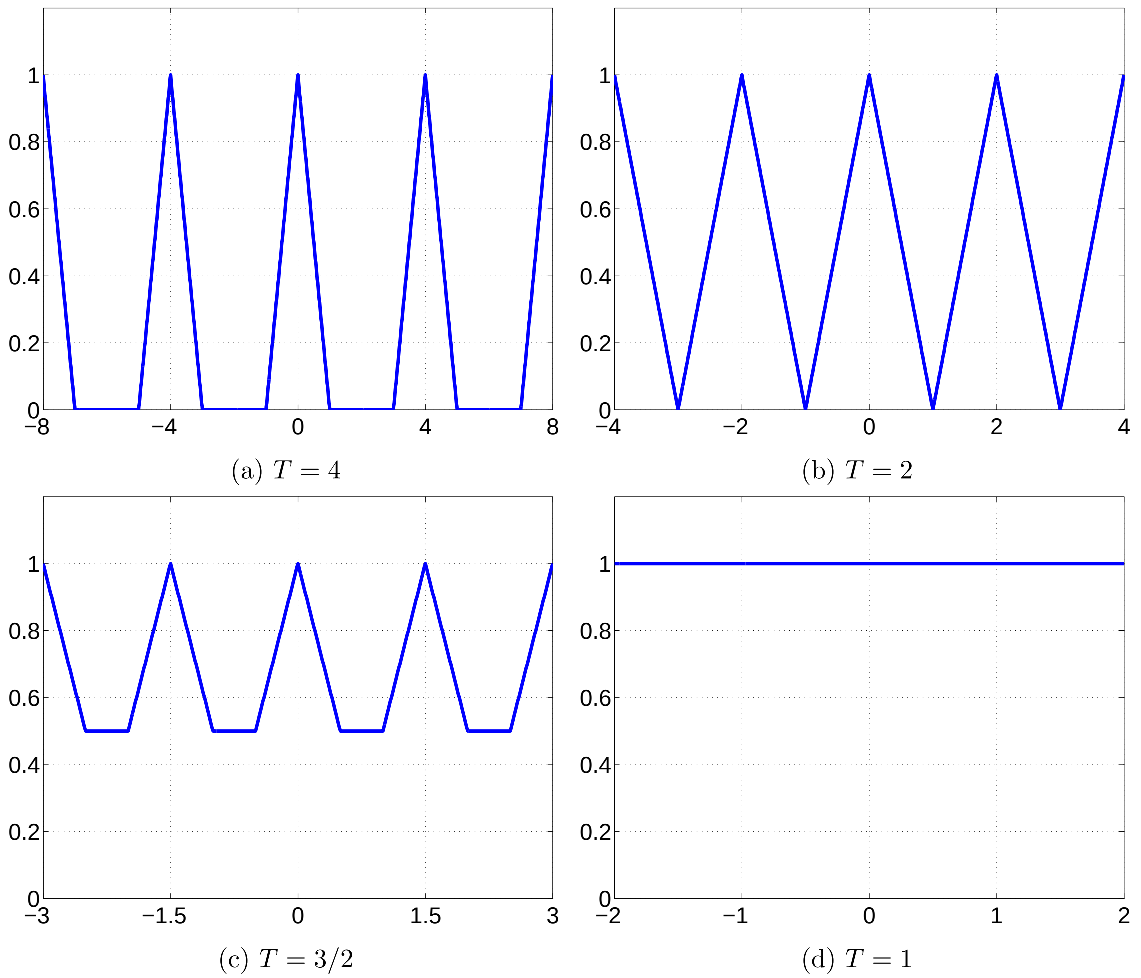

Figure 3: OW 2.23

7. (OW 2.28 (a)-(d), (g))

Solution

(a) Causal and stable

$$h[n] = 0 \qquad n < 0 \Rightarrow \text{casual}$$
$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{5}\right)^n u[n] = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1-(1/5)} = \frac{5}{4} < \infty \Rightarrow \text{stable}$$

(b) Not causal and stable

$$h[n] \neq 0 \qquad n = -1, -2 \Rightarrow \text{not casual}$$
$$\sum_{n=-2}^{\infty} (0.8)^n \, 5 < \infty \Rightarrow \text{stable}$$

(c) Not causal (anti-causal) and unstable

$$h[n] = 0 \qquad n > 0 \Rightarrow \text{not casual, actually anti-causal}$$
$$\sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^n = \infty \Rightarrow \text{not stable}$$

(d) Not causal and stable

$$h[n] \neq 0 \qquad n < 3 \Rightarrow \text{not causal}$$
$$\sum_{n=-\infty}^{3} (5)^n = \sum_{n=-3}^{\infty} \left(\frac{1}{5}\right)^n = \frac{\frac{1}{5}^{-3}}{1-1/5} = \frac{625}{4} < \infty \Rightarrow \text{stable}$$

(i) Causal and stable

$$h[n] = 0 \qquad n < 1 \Rightarrow \text{casual}$$
$$\sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^n \approx \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n < \infty \Rightarrow \text{stable}$$

Note: the geometric term $\left(\frac{1}{3}\right)^n$ reduces much faster than the n term, hence convergence.

8. (OW 2.29)

Solution

(a) Causal and stable

$$h(t) = 0 \qquad \forall t < 2 \Rightarrow \text{casual}$$
$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{2}^{\infty} e^{-4t} dt = -\frac{1}{4} \left[e^{-4t}\right]_2^{\infty} = \frac{1}{4} e^{-8} < \infty \Rightarrow \text{stable}$$

(b) Not causal and not stable

$$h(t) = 0 \qquad \forall t > 3 \Rightarrow \text{not casual}$$
$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{3} e^{-6t} dt = -\frac{1}{6} \left[e^{-6t}\right]_{-\infty}^{3} = \frac{1}{6} [\infty - e^{-18}] \Rightarrow \text{not stable}$$

(c) Not causal and stable

$$h(t) = 0 \qquad \forall t < -50 \Rightarrow \text{not casual}$$
$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-50}^{\infty} e^{-2t}dt = -\frac{1}{2}\left[e^{-2t}\right]_{-50}^{\infty} = \frac{1}{2}e^{100} < \infty \Rightarrow \text{stable}$$

(d) Not causal and stable

$$h(t) = 0 \qquad \forall t > -1 \Rightarrow \text{not casual}$$
$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{-1} e^{2t}dt = \frac{1}{2}\left[e^{2t}\right]_{-\infty}^{-1} = \frac{1}{2}[e^{-2} - 0] < \infty \Rightarrow \text{stable}$$

(e) Not causal and stable

$$h(t) \text{ defined } \forall t \Rightarrow \text{not casual}$$
$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{0} e^{6t}dt + \int_{0}^{\infty} e^{-6t}dt = 2\int_{0}^{\infty} e^{-6t}dt = 2\frac{1}{6} = \frac{1}{3} < \infty \Rightarrow \text{stable}$$

(f) Causal and stable

$$h(t) = 0 \qquad \forall t < 0 \Rightarrow \text{casual}$$
$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{0}^{\infty} te^{-t}dt = \left[\frac{e^{-t}}{1^2}(-t-1)\right]_{0}^{\infty} = [0 - (1(-1))] = 1 < \infty \Rightarrow \text{stable}$$

(g) Causal and not stable

$$h(t) = 0 \qquad \forall t < 0 \Rightarrow \text{casual}$$
$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{0}^{\infty} 2e^{-t}dt + \int_{0}^{\infty} e^{(t-100)/100}dt = -2\left[e^{-t}\right]_{0}^{\infty} + 100e^{-100}\left[e^{t/100}\right]_{0}^{\infty}$$
$$= 2 + 100e^{-100}[\infty - 1] \Rightarrow \text{not stable}$$

9. (OW 2.40)

Solution

(a)

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau-2)d\tau$$
$$\text{let } t' = \tau - 2 \Rightarrow \tau = t' + 2$$
$$= \int_{-\infty}^{t-2} e^{-(t-t'-2)}x(t')dt'$$
$$\Rightarrow h(t) = e^{(-t-2)}u(t-2)$$

(b)

$$t < 1 \qquad y(t) = 0$$
$$1 \leq t < 4 \qquad y(t) = \int_{2}^{t+1} e^{-(\tau-2)}d\tau = -1\left[e^{-(\tau-2)}\right]_{2}^{t+1} = 1 - e^{-(t-1)}$$
$$t \geq 4 \qquad y(t) = \int_{t-2}^{t+1} e^{-(\tau-2)}d\tau = -1\left[e^{-(\tau-2)}\right]_{t-2}^{t+1} = e^{-(t-4)} - e^{-(t-1)}$$

$$y(t) = \begin{cases} 0 & t < 1 \\ 1 - e^{-(t-1)} & 1 \leq t < 4 \\ e^{-(t-4)} - e^{-(t-1)} & t \geq 4 \end{cases}$$

10. (OW 2.46)

Solution

$$x(t) = 2e^{-3t}u(t-1) \longrightarrow y(t)$$
$$\frac{dx(t)}{dt} \longrightarrow -3y(t) + e^{-2t}u(t)$$

$$\begin{aligned} \frac{dx(t)}{dt} &= 2e^{-3t}\delta(t-1) + -6e^{-3t}u(t-1) \\ &= 2e^{-3}\delta(t-1) - 6e^{-3t}u(t-1) \\ &= 2e^{-3}\delta(t-1) - 3x(t) \end{aligned}$$

$$\Rightarrow -3x(t) + 2e^{-3}\delta(t-1) \longrightarrow -3y(t) + e^{-2t}u(t)$$

since $-3x(t) \longrightarrow -3y(t)$ by definition and linearity

$$\underbrace{2e^{-3}}_{a}\delta(t-1) \longrightarrow e^{-2}u(t)$$
$$a\delta(t-1) \longrightarrow e^{-2t}u(t) = ah(t-1)$$

$$\begin{aligned} \Rightarrow h(t-1) &= \frac{1}{a}e^{-2t}u(t) = \frac{1}{2e^{-3}}e^{-2t}u(t) \\ &= \frac{1}{2}e^{-2t+3}u(t) \end{aligned}$$

let $\tau = t - 1$

$$\begin{aligned} h(\tau) &= \frac{1}{2}e^{-2(\tau+1)+3}u(\tau+1) \\ &= \frac{1}{2}e^{-2\tau+1}u(\tau+1) \end{aligned}$$