

Homework #4
Due Su. 2/25

Note: The **Basic Problems with Answers** will be worth half as much as the other questions. You must show all your work to receive credit.

1. (OW 3.21)

Solution

Given the following information about the signal

$$T = 8, \quad a_1 = a_{-1}^* = j, \quad a_5 = a_{-5} = 2,$$

the periodic signal $x(t)$ can be expressed using the complex exponential form of the Fourier Series and converted using $\omega_0 = 2\pi/T = \pi/4$.

$$\begin{aligned} x(t) &= a_{-5}e^{-j5\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_1e^{j\omega_0 t} + a_5e^{j5\omega_0 t} \\ &= 2[e^{-j5\omega_0 t} + e^{j5\omega_0 t}] + j[e^{j\omega_0 t} - e^{-j\omega_0 t}] \\ &= 4\cos(5\omega_0 t) - 2\sin(\omega_0 t) \\ &= 4\cos\left(\frac{5\pi}{4}t\right) - 2\cos\left(\frac{\pi}{4}t - \frac{\pi}{2}\right) \end{aligned}$$

2. (OW 3.24)

Solution

(a) Find the average signal value over a period

$$a_0 = \frac{1}{T} \int_T x(t) dt = \frac{1}{2} \left[\frac{1}{2}(1)(2) \right] = \frac{1}{2}.$$

(b) Define

$$g(t) = \frac{dx(t)}{dt} = \begin{cases} 1 & 0 \leq t \leq 1 \\ -1 & 1 \leq t \leq 2 \end{cases}.$$

Then FS coefficients can be found by noting

$$g(t) \longrightarrow b_k = \frac{1}{T} \int_T g(t) e^{-jk\omega_0 t} dt.$$

$$k = 0 \quad b_0 = 0 \quad \text{average over single period}$$

$$\begin{aligned} k \neq 0 \quad b_k &= \frac{1}{2} \left[\int_0^1 e^{-jk\omega_0 t} dt - \int_1^2 e^{-jk\omega_0 t} dt \right] \\ &= \frac{1}{2} \left[\frac{1}{-jk\omega_0} e^{-jk\omega_0 t} \right]_0^1 - \frac{1}{2} \left[\frac{1}{-jk\omega_0} e^{-jk\omega_0 t} \right]_1^2 \\ &= \frac{1}{2jk\omega_0} [1 - e^{-j\omega_0}] + \frac{1}{2jk\omega_0} [e^{-j2\omega_0} - e^{-j\omega_0}] \\ &= \frac{1}{2jk\omega_0} [1 - 2e^{-j\omega_0} + e^{-j2\omega_0}] \\ &= \frac{1}{2jk\pi} \left[1 - 2e^{-jk\pi} + \underbrace{e^{-jk2\pi}}_{=1} \right] \\ &= \frac{1}{jk\pi} [1 - e^{-jk\pi}] \end{aligned}$$

(c) Using the differentiation property

$$\begin{aligned} x(t) &\longleftrightarrow a_k \\ \frac{dx(t)}{dt} &\longleftrightarrow b_k = jk\omega_0 a_k = jk\pi a_k \\ \Rightarrow a_k &= \frac{1}{jk\pi} b_k = \frac{1}{(jk\pi)^2} [1 - e^{-j\pi}] = \frac{1}{k^2\pi^2} [e^{-j\pi} - 1] \end{aligned}$$

therefore,

$$a_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{k^2\pi^2} [e^{-j\pi} - 1] & \text{else} \end{cases}.$$

3. (OW 3.25)

Solution

(a)

$$\begin{aligned} x(t) &= \frac{1}{2} [e^{j4\pi t} + e^{-j4\pi t}] \\ \Rightarrow a_k &\longrightarrow a_1 = a_{-1} = \frac{1}{2} \end{aligned}$$

(b)

$$\begin{aligned} y(t) &= \frac{1}{2j} [e^{j4\pi t} - e^{-j4\pi t}] \\ \Rightarrow b_k &\longrightarrow b_1 = \frac{1}{2j}, \quad b_{-1} = -\frac{1}{2j} \end{aligned}$$

(c) The multiplicative property states the product in the time domain has convolution of the FS coefficients.

$$z(t) = x(t)y(t) \longleftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

$$\begin{aligned} c_k &= \left[\frac{1}{2} \delta(k+1) + \frac{1}{2} \delta(k-1) \right] * \left[-\frac{1}{2j} \delta(k+1) + \frac{1}{2j} \delta(k-1) \right] \\ &= -\frac{1}{4j} \delta(k+2) + \frac{1}{4j} \delta(k) - \frac{1}{4j} \delta(k) + \frac{1}{4j} \delta(k-2) \\ &= \underbrace{-\frac{1}{4j} \delta(k+2)}_{c_{-2}} + \underbrace{\frac{1}{4j} \delta(k-2)}_{c_2} \end{aligned}$$

(d) Using pure trigonometric expansion

$$\begin{aligned} z(t) &= \cos(4\pi t) \sin(4\pi t) \\ &= \frac{1}{2} [e^{j4\pi t} + e^{-j4\pi t}] \cdot \frac{1}{2j} [e^{j4\pi t} - e^{-j4\pi t}] \\ &= \frac{1}{4j} [e^{j2(4\pi)t} - e^0 + e^0 - e^{-j2(4\pi)t}] \\ &= \underbrace{\frac{1}{4j} e^{j2(4\pi)t}}_{c_2} - \underbrace{\frac{1}{4j} e^{-j2(4\pi)t}}_{c_{-2}} \end{aligned}$$

4. (OW 3.27)

Solution

$$N = 5, \quad a_0 = 2, \quad a_2 = a_{-2}^* = 2e^{j\pi/6}, \quad a_4 = a_{-4}^* = e^{j\pi/3}, \quad \omega_0 = \frac{2\pi}{5}$$

$$\begin{aligned} x[n] &= 2 + a_2 e^{j2\omega_0 n} + a_{-2} e^{-j2\omega_0 n} + a_4 e^{j4\omega_0 n} + a_{-4} e^{-j4\omega_0 n} \\ &= 2 + 2e^{j\pi/6} e^{j2\omega_0 n} + 2e^{-j\pi/6} e^{-j2\omega_0 n} + e^{j\pi/3} e^{j4\omega_0 n} + e^{-j\pi/3} e^{-j4\omega_0 n} \\ &= 2 + 2 \left[e^{j(2\omega_0 n + \pi/6)} + e^{-j(2\omega_0 n + \pi/6)} \right] + \left[e^{j(4\omega_0 n + \pi/3)} + e^{-j(4\omega_0 n + \pi/3)} \right] \\ &= 2 + 4 \cos(2\omega_0 n + \pi/6) + 2 \cos(4\omega_0 n + \pi/3) \\ &= 2 + 4 \sin(2\omega_0 n + \pi/6 + \pi/2) + 2 \sin(4\omega_0 n + \pi/3 + \pi/2) \\ &= 2 + 4 \sin(2\omega_0 n + 2\pi/3) + 2 \sin(4\omega_0 n + 5\pi/6) \end{aligned}$$

5. (OW 3.33)

Solution

$$\frac{\partial}{\partial t} y(t) + 4y(t) = x(t)$$

assume input of form $e^{j\omega t}$ and find frequency response

$$e^{j\omega t} \longrightarrow H(j\omega) e^{j\omega t} \text{ for LTI system}$$

$$\Rightarrow \frac{\partial}{\partial t} y(t) = j\omega H(j\omega) e^{j\omega t}$$

$$\Rightarrow j\omega H(j\omega) e^{j\omega t} + 4H(j\omega) e^{j\omega t} = e^{j\omega t}$$

$$\Rightarrow H(j\omega) [j\omega + 4] = 1$$

$$H(j\omega) = \frac{1}{4 + j\omega} \quad \leftarrow \text{use this for } y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$(a) \quad \omega_0 = 2\pi, T = 1$$

$$x(t) = \cos(2\pi t) = \frac{1}{2} [e^{j2\pi t} + e^{-j2\pi t}]$$

$$\begin{aligned} y(t) &= \frac{1}{2} [H(j2\pi) e^{j2\pi t} + H(-j2\pi) e^{-j2\pi t}] \\ &= \frac{1}{2} \left[\frac{1}{4 + j2\pi} e^{j2\pi t} + \frac{1}{4 - j2\pi} e^{-j2\pi t} \right] \end{aligned}$$

$$\Rightarrow b_1 = \frac{1}{2} \left(\frac{1}{4 + j2\pi} \right), \quad b_{-1} = \frac{1}{2} \left(\frac{1}{4 - j2\pi} \right)$$

(b)

$$\begin{aligned}
x(t) &= \sin(\underbrace{4\pi t}_{T_1=1/2}) + \cos(\underbrace{6\pi t}_{T_2=1/3} + \pi/4) \\
&\Rightarrow T = 1, \omega_0 = 2\pi \\
&= \frac{1}{2j} [e^{j2\omega_0 t} - e^{-j2\omega_0 t}] + \frac{1}{2} e^{j\pi/4} e^{j3\omega_0 t} + \frac{1}{2} e^{-j\pi/4} e^{-j3\omega_0 t}
\end{aligned}$$

$$y(t) = \frac{1}{2j} [H(j4\pi)e^{j2\omega_0 t} - H(-j4\pi)e^{-j2\omega_0 t}] + \frac{1}{2} e^{j\pi/4} H(j6\pi)e^{j3\omega_0 t} + \frac{1}{2} e^{-j\pi/4} H(-j6\pi)e^{-j3\omega_0 t}$$

$$\begin{aligned}
\Rightarrow b_2 &= \frac{1}{2j} \left(\frac{1}{4 + j4\pi} \right), & b_{-2} &= -\frac{1}{2j} \left(\frac{1}{4 - j4\pi} \right), \\
b_3 &= \frac{1}{2j} e^{j\pi/4} \left(\frac{1}{4 + j6\pi} \right), & b_{-3} &= \frac{1}{2j} e^{-j\pi/4} \left(\frac{1}{4 - j6\pi} \right)
\end{aligned}$$

6. (OW 3.36)

Solution

Similar to 3.33

$$\begin{aligned}
y[n] - \frac{1}{4}y[n-1] &= x[n] \\
y[n] &= H(e^{j\omega})e^{j\omega n}, \quad y[n-1] = H(e^{j\omega})e^{j\omega(n-1)} \\
H(e^{j\omega}) \left[1 - \frac{1}{4}e^{-j\omega} \right] e^{j\omega n} &= e^{j\omega n} \\
H(e^{j\omega}) &= \frac{1}{1 - \frac{1}{4}e^{-j\omega}}
\end{aligned}$$

(a)

$$\begin{aligned}
x[n] &= \sin\left(\frac{3\pi}{4}n\right), \quad N_0 = 8 \Rightarrow \omega_0 = \frac{2\pi}{8} = \frac{\pi}{4} \\
&= \frac{1}{2j} [e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n}]
\end{aligned}$$

$$\begin{aligned}
y[n] &= \frac{1}{2j} \left[H\left(e^{j\frac{3\pi}{4}}\right) e^{j\frac{3\pi}{4}n} - H\left(e^{-j\frac{3\pi}{4}}\right) e^{-j\frac{3\pi}{4}n} \right] \\
&= \frac{1}{2j} \left[\frac{1}{1 - \frac{1}{4}e^{-j\frac{3\pi}{4}}} e^{j\frac{3\pi}{4}n} - \frac{1}{1 - \frac{1}{4}e^{j\frac{3\pi}{4}}} e^{-j\frac{3\pi}{4}n} \right] \\
\Rightarrow b_3 &= \frac{1}{2j} \left(\frac{1}{1 - \frac{1}{4}e^{-j\frac{3\pi}{4}}} \right), \quad b_{-3} = \left(\frac{1}{2j} \frac{1}{1 - \frac{1}{4}e^{j\frac{3\pi}{4}}} \right)
\end{aligned}$$

(b)

$$x[n] = \underbrace{\cos\left(\frac{\pi}{4}n\right)}_{N_1=8} + 2 \underbrace{\cos\left(\frac{\pi}{2}n\right)}_{N_2=4}.$$

Therefore the period of $x[n]$ is $N = 8 \Rightarrow w_0 = \frac{2\pi}{8} = \frac{\pi}{4}$.

Following the same procedure as above

$$\begin{aligned} x[n] &= \frac{1}{2}[e^{j\pi/4n} + e^{-j\pi/4n}] + [e^{j2\pi/4n} + e^{-j2\pi/4n}] \\ y[n] &= \frac{1}{2}[\underbrace{H(e^{j\pi/4})}_{k=1}e^{j\pi/4n} + \underbrace{H(e^{-j\pi/4})}_{k=-1}e^{-j\pi/4n}] + [\underbrace{H(e^{j2\pi/4})}_{k=2}e^{j2\pi/4n} + \underbrace{H(e^{-j2\pi/4})}_{k=-2}e^{-j2\pi/4n}] \end{aligned}$$

$$\begin{aligned} \Rightarrow b_1 &= \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}e^{-j\pi/4}} \right), & b_2 &= \left(\frac{1}{1 - \frac{1}{4}e^{-j\pi/2}} \right), \\ b_{-1} &= \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}e^{j\pi/4}} \right), & b_{-2} &= \left(\frac{1}{1 - \frac{1}{4}e^{j\pi/2}} \right) \end{aligned}$$