Homework #3
Due Su 09/28

Note:

OW	Oppenheim and Wilsky
SSS	Schaum's Signals and Systems
SPR	Schaum's Probability, Random Variables, and Random Processes

Be sure to show all your work for credit.

1. (SSS 5.75)

Solution

First find the differential relationship between input and output by KVL

$$x(t) = Ri(t) + v_c(t) \qquad i(t) = C\frac{dv_c(t)}{dt} \qquad v_c(t) = x(t) - y(t).$$

Substituting variables results in

$$\begin{aligned} x(t) &= RC\frac{dv_c(t)}{dt} + v_c(t) \\ &= RC\frac{d}{dt}\left(x(t) - y(t)\right) + \left(x(t) - y(t)\right). \end{aligned}$$

Giving the final differential equation

$$RC\frac{d}{dt}y(t) + y(t) = RC\frac{d}{dt}x(t).$$

The frequency response is found by taking the FT of both sides

$$\begin{aligned} Y(j\omega)\left[RCj\omega + 1\right] &= RCj\omega X(j\omega) \\ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} &= \frac{RCj\omega}{RCj\omega + 1} = \frac{j\omega}{j\omega + 1/RC}. \end{aligned}$$

You can tell what type of filter this is by examining the magnitude (squared) response.

$$\begin{aligned} |H(j\omega)|^2 &= H(j\omega)H^*(j\omega) \\ &= \left(\frac{j\omega}{j\omega + 1/RC}\right)\left(\frac{-j\omega}{-j\omega + 1/RC}\right) = \frac{-j^2\omega^2}{-j^2\omega^2 + (1/RC)^2} = \frac{\omega^2}{(1/RC)^2 + \omega^2}. \end{aligned}$$

Notice this is a highpass filter because as $\omega \to 0$, $|H(j\omega)|^2 \to 0$ and $\omega \to \infty$, $|H(j\omega)|^2 \to 1$.

2. (OW 4.21 (a)-(d), (f))

Solution

(a) Using the FT multiplication property

$$\begin{aligned} e^{-\alpha}u(t)\cos(\omega_0 t) &\leftrightarrow \frac{1}{2\pi}\left[\frac{1}{\alpha + j\omega}\right] * \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \\ &= \frac{1}{2}\left[\frac{1}{\alpha + j\omega}\right] * \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \\ &= \frac{1}{2}\left[\frac{1}{\alpha + j(\omega - \omega_0)} + \frac{1}{\alpha + j(\omega + \omega_0)}\right] \end{aligned}$$

(b)

$$x(t) = e^{-3|t|}\sin(2t) = \left[e^{-3t}u(t) + e^{3t}u(-t)\right]\sin(2t)$$

using the same procedure as from (a)

$$\begin{aligned} e^{-3t}u(t)\sin(2t) &\leftrightarrow \frac{1}{2}\left[\frac{1}{3+j\omega}\right] * \frac{\pi}{j}\left[\delta(\omega-2)+\delta(\omega+2)\right] \\ &= \frac{1}{2j}\left[\frac{1}{\alpha+j(\omega-2)} + \frac{1}{\alpha+j(\omega+2)}\right] \\ &= X_1(j\omega) \end{aligned}$$

using the time reversal property

$$\begin{aligned} e^{3t}u(-t)\sin(2t) &\leftrightarrow \frac{1}{2}\left[\frac{1}{3-j\omega}\right] * \frac{\pi}{j}\left[\delta(\omega-2)+\delta(\omega+2)\right] \\ &= \frac{1}{2j}\left[\frac{1}{\alpha-j(\omega+2)} + \frac{1}{\alpha-j(\omega-2)}\right] \\ &= X_2(j\omega) \\ X(j\omega) &= X_1(j\omega) + X_2(j\omega) \end{aligned}$$

(c) Through direct evaluation of integral

$$\begin{aligned} X(j\omega) &= \int_{-1}^{1}(1+\cos(\pi t))e^{-j\omega t}dt \\ &= \int_{-1}^{1} e^{-j\omega t}dt + \int_{-1}^{1}\frac{1}{2}e^{-j(\omega+\pi)t}dt + \int_{-1}^{1}\frac{1}{2}e^{-j(\omega-\pi)t}dt \\ &= \frac{2\sin(\omega t)}{\omega} + \frac{1}{\omega+\pi}\sin(\omega+\pi) + \frac{1}{\omega-\pi}\sin(\omega-\pi) \\ &= \frac{2\sin(\omega t)}{\omega} - \frac{\sin(\omega)}{\omega+\pi} - \frac{\sin(\omega)}{\omega-\pi} \end{aligned}$$

(d)

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty}\sum_{k=0}^{\infty}\alpha^k\delta(t-kT)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\sum_{k=0}^{\infty}\alpha^k\delta(t-kT)e^{j\omega kT}dt \\ &= \sum_{k=0}^{\infty}\alpha^k e^{j\omega kT}\int_{-\infty}^{\infty}\delta(t-kT)dt = \sum_{k=0}^{\infty}(\alpha e^{j\omega T})^k \\ &= \frac{1}{1-\alpha e^{j\omega T}} \end{aligned}$$

(f) Notice this is the product of two sinc signals.

$$x_1(t) = \frac{\sin\pi t}{\pi t} \leftrightarrow X_1(j\omega) = \begin{cases} 1 & |\omega| < \pi \\ 0 & \text{else.} \end{cases}$$

$$x_2(t) = \frac{\sin 2\pi(t-1)}{\pi(t-1)} \leftrightarrow X_2(j\omega) = \begin{cases} e^{-j\omega} & |\omega| < 2\pi \\ 0 & \text{else.} \end{cases}$$

The FT of the signal can be found by convolution of the two sinc signals in the frequency domain

$$X(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$
$$= \begin{cases} 0 & -\pi \leq \omega \leq \pi \\ \frac{1}{j2\pi}\left[e^{-j\omega}+1\right] & -3\pi \leq \omega < -\pi \\ -\frac{1}{j2\pi}\left[e^{-j\omega}+1\right] & \pi < \omega \leq 3\pi \\ 0 & \text{else} \end{cases}.$$

One of the cases is found as

$$X(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(j\theta)X_1(j(\omega-\theta))d\theta$$
$$= \frac{1}{2\pi}\int_{\omega-\pi}^{2\pi} e^{-j\theta}d\theta = \frac{1}{-2\pi j}\left[e^{-j\theta}\right]_{\omega-\pi}^{2\pi}$$
$$= \frac{1}{-2\pi j}\left[e^{-j2\pi} - e^{-j\omega}e^{j\pi}\right]$$
$$= \frac{1}{-2\pi j}\left[1+e^{-j\omega}\right].$$

3. (OW 4.22 (a)-(d)) + (SSS 5.69)

Solution

(a)

$$X(j\omega) = \frac{2\sin[3(\omega-2\pi)]}{(\omega-2\pi)}$$
$$= X_1(j(\omega-2\pi)) \qquad \text{with} \qquad X_1(j\omega) = \frac{2\sin(3\omega)}{\omega} \leftrightarrow x_1(t) = \begin{cases} 1 & |t|<3 \\ 0 & \text{else} \end{cases}$$
$$X(j\omega) \leftrightarrow e^{j2\pi t}x_1(t)$$

therefore

$$x(t) = \begin{cases} e^{j2\pi t} & |t|<3 \\ 0 & \text{else} \end{cases}$$

(b) Using Euler's relationship

$$x(t) = \frac{1}{2}e^{j\pi/3}\delta(t+4) + \frac{1}{2}e^{-j\pi/3}\delta(t-4)$$

(c) The inverse Fourier transform equation can be re-written as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|e^{j\angle X(j\omega)}e^{j\omega t}d\omega$$

Compute two integrals between [-1,0] and [0,1] using integration by parts to find

$$x(t) = \frac{\sin(t-3)}{\pi(t-3)} + \frac{\cos(t-3)-1}{\pi(t-3)^2}$$

(d)

$$x(t) = \frac{2j}{\pi}\sin(t) + \frac{3}{\pi}\cos(2\pi t)$$

(e) (SSS 5.69)

$$\begin{aligned}
X(j\omega) &= \frac{1}{2-\omega^2+3j\omega} = \frac{1}{2+j^2\omega^2+3j\omega} = \frac{1}{(2+j\omega)(1+j\omega)} \\
&= \frac{A}{2+j\omega} + \frac{B}{1+j\omega} \\
A &= [X(j\omega)(2+j\omega)]_{j\omega=-2} = -1 \\
B &= [X(j\omega)(1+j\omega)]_{j\omega=-1} = 1 \\
X(j\omega) &= \frac{-1}{2+j\omega} + \frac{1}{1+j\omega}
\end{aligned}$$

Taking the inverse transform results in

$$x(t) = -e^{2t}u(t) + e^{-t}u(t)$$

4. (OW 4.23)

Solution

$$\begin{aligned}
x_0 &= \begin{cases} e^{-t} & 0 \le t \le 1 \\ 0 & \text{else} \end{cases} \\
X_0(j\omega) &= \int_0^1 e^{-t}e^{-j\omega t}dt = \int_0^1 e^{-(j\omega+1)t}dt \\
&= \frac{1}{-(j\omega+1)}\left[e^{-(j\omega+1)t}\right]_0^1 = \frac{1}{j\omega+1}\left[1-e^{-(j\omega+1)}\right]
\end{aligned}$$

(a)

$$x_1(t) = x_0(-t) + x_0(t) \longleftrightarrow X_1(j\omega) = X_0(-j\omega) + X_0(j\omega)$$

$$X_1(j\omega) = \frac{1}{-j\omega+1}\left[1-e^{j\omega-1}\right] + \frac{1}{j\omega+1}\left[1-e^{-j\omega-1}\right]$$

(b)

$$x_2(t) = -x_0(-t) + x_0(t) \longleftrightarrow X_2(j\omega) = -X_0(-j\omega) + X_0(j\omega)$$

$$X_2(j\omega) = \frac{1}{j\omega-1}\left[1-e^{j\omega-1}\right] + \frac{1}{j\omega+1}\left[1-e^{-j\omega-1}\right]$$

(c)

$$x_3(t) = x_0(t+1) + x_0(t) \longleftrightarrow X_3(j\omega) = e^{j\omega}X_0(j\omega) + X_0(j\omega) = X_0(j\omega)\left[e^{j\omega}+1\right]$$

$$X_3(j\omega) = \frac{1}{j\omega+1}\left[1-e^{-j\omega-1}\right]\left[e^{j\omega}+1\right] = \frac{1}{j\omega+1}\left[1+e^{j\omega}-e^{-1}(1+e^{-j\omega})\right]$$

(d)

$$x_4(t) = tx_0(t) \longleftrightarrow X_4(j\omega) = j\frac{d}{d\omega}X_0(j\omega)$$

$$\begin{aligned} X_4(j\omega) &= j\left[(j\omega+1)^{-1}\{(j\omega+1)e^{-(j\omega+1)}\} + \frac{-1}{(j\omega+1)^2}\left[1 - e^{-(j\omega+1)}\right]\right] \\ &= j\left[e^{-(j\omega+1)} - \frac{1}{(j\omega+1)^2}\left[1 - e^{-(j\omega+1)}\right]\right] \end{aligned}$$

5. (OW 4.27)

Solution

$$\begin{aligned} x(t) = u(t-1) - 2u(t-2) + u(t-3) &\longleftrightarrow X(j\omega) \\ \tilde{x}(t) = \sum_k x(t-kT) &\longleftrightarrow a_k \end{aligned}$$

(a) Let

$$x(t) = x_0(t-1.5) - x_0(t-2.5)$$

where

$$x_0(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & \text{else} \end{cases} \longleftrightarrow X_0(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

Then

$$\begin{aligned} X(j\omega) &= e^{-j\omega(3/2)}X_0(j\omega) - e^{-j\omega(5/2)}X_0(j\omega) = X_0(j\omega)e^{-j\omega(3/2)}[1 - e^{-j\omega}] \\ &= \frac{2\sin(\omega/2)}{\omega}e^{-j\omega(3/2)}[1 - e^{-j\omega}] \end{aligned}$$

(b) Using the same technique as in (a) above define

$$\tilde{x}_0(t) \longleftrightarrow b_k = \frac{\sin(k\omega/2)}{k\pi}$$

Then

$$\begin{aligned} a_k &= \mathcal{FS}\{\tilde{x}_0(t-1.5)\} - \mathcal{FS}\{\tilde{x}_0(t-2.5)\} \\ &= e^{-jk\omega_0(3/2)}b_k - e^{-jk\omega_0(5/2)}b_k = b_k e^{-jk\omega_0(3/2)}[1 - e{-}jk\omega_0] \\ &= \frac{\sin(k\omega/2)}{k\pi}e^{-jk\omega_0(3/2)}[1 - e{-}jk\omega_0] \end{aligned}$$

(c) Using $T = \frac{2\pi}{\omega_0} \rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$

$$\begin{aligned} \frac{1}{T}X(j\frac{2\pi k}{T}) &= \frac{\omega_0}{2\pi}X(j\omega_0 k) \\ &= \frac{\omega_0}{2\pi}\frac{2\sin(\omega_0 k/2)}{\omega_0 k}e^{-j\omega_0 k(3/2)}[1 - e^{-j\omega_0 k}] \\ &= \frac{\sin(\omega_0 k/2)}{\pi k}e^{-j\omega_0 k(3/2)}[1 - e^{-j\omega_0 k}] \\ &= a_k \end{aligned}$$

6. (OW 4.34)

Solution

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} = \frac{j\omega + 4}{(j\omega + 3)(j\omega + 2)} = \frac{Y(j\omega)}{X(j\omega)}$$

(a) Cross multiply $H(j\omega)$ and use the differentiation property

$$Y(j\omega)[6 + (j\omega)^2 + 5j\omega] = X(j\omega)[j\omega + 4]$$
$$6y(t) + \frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) = \frac{d}{dt}x(t) + 4x(t)$$

(b) Find $h(t)$ by finding partial fraction expansion of $H(j\omega)$ and inverse Fourier transform

$$H(j\omega) = \frac{j\omega + 4}{(j\omega + 3)(j\omega + 2)} = \frac{A}{(j\omega + 3)} + \frac{B}{(j\omega + 2)}$$
$$A = [(j\omega + 3)H(j\omega)]_{j\omega=-3} = \left[\frac{j\omega + 4}{j\omega + 2}\right]_{j\omega=-3} = \frac{1}{-1} = -1$$
$$B = [(j\omega + 2)H(j\omega)]_{j\omega=-2} = \left[\frac{j\omega + 4}{j\omega + 3}\right]_{j\omega=-2} = \frac{2}{1} = 2$$
$$H(j\omega) = \frac{-1}{(j\omega + 3)} + \frac{2}{(j\omega + 2)}$$

taking inverse Fourier Transform

$$h(t) = -e^{-3t}u(t) + 2e^{-2t}u(t)$$

(c) Use the convolution property of FT, partial fraction expansion, and iFT

$$Y(j\omega) = H(j\omega)X(j\omega)$$
$$= \frac{j\omega + 4}{(j\omega + 3)(j\omega + 2)}\left[\frac{1}{j\omega + 4} - \frac{1}{(j\omega + 4)^2}\right] = \frac{j\omega + 4}{(j\omega + 3)(j\omega + 2)}\left[\frac{j\omega + 3}{(j\omega + 4)^2}\right]$$
$$= \frac{1}{(j\omega + 2)(j\omega + 4)} = \frac{A}{(j\omega + 2)} + \frac{B}{(j\omega + 4)}$$
$$A = [(j\omega + 2)Y(j\omega)]_{j\omega=-2} = \left[\frac{1}{j\omega + 4}\right]_{j\omega=-2} = \frac{1}{2}$$
$$B = [(j\omega + 4)Y(j\omega)]_{j\omega=-4} = \left[\frac{1}{j\omega + 2}\right]_{j\omega=-4} = \frac{1}{-2}$$
$$Y(j\omega) = \frac{1/2}{(j\omega + 2)} - \frac{1/2}{(j\omega + 4)}$$
$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t)$$

7. (OW 4.36)

Solution

(a)

$$X(j\omega) = \frac{1}{j\omega + 1} + \frac{1}{j\omega + 3} = \frac{j\omega + 3 + j\omega + 1}{(j\omega + 1)(j\omega + 3)} = \frac{2j\omega + 4}{(j\omega + 1)(j\omega + 3)}$$
$$Y(j\omega) = \frac{2}{j\omega + 1} - \frac{2}{j\omega + 4} = 2\frac{3}{(j\omega + 4)(j\omega + 1)}$$
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = 2\frac{3}{(j\omega + 4)(j\omega + 1)}\left[\frac{(j\omega + 1)(j\omega + 3)}{2(j\omega + 2)}\right] = \frac{3(j\omega + 3)}{(j\omega + 4)(j\omega + 2)}$$

(b)

$$
\begin{aligned}
H(j\omega) &= \frac{3(j\omega+3)}{(j\omega+4)(j\omega+2)} = \frac{A}{(j\omega+4)} + \frac{B}{(j\omega+2)} \\
A &= [(j\omega+4)H(j\omega)]_{j\omega=-4} = \left[\frac{3(j\omega+3)}{j\omega+2}\right]_{j\omega=-4} = \frac{-3}{-2} = \frac{3}{2} \\
B &= [(j\omega+2)H(j\omega)]_{j\omega=-2} = \left[\frac{3(j\omega+3)}{j\omega+4}\right]_{j\omega=-2} = \frac{3}{2} \\
H(j\omega) &= \frac{3/2}{(j\omega+4)} + \frac{3/2}{(j\omega+2)} \\
h(t) &= \frac{3}{2}\left[e^{-4t} + e^{-2t}\right]u(t)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{Y(j\omega)}{X(j\omega)} &= \frac{3(j\omega+3)}{(j\omega+4)(j\omega+2)} = \frac{3j\omega+9}{(j\omega)^2+6j\omega+8} \\
Y(j\omega)[(j\omega)^2+6j\omega+8] &= X(j\omega)[3j\omega+9] \\
\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) &= 3\frac{d}{dt}x(t) + 9x(t)
\end{aligned}
$$

8. (OW 4.44)

Solution

$$
z(t) = e^{-t}u(t) + 3\delta(t) \longleftrightarrow Z(j\omega) = \frac{1}{j\omega+1} + 3
$$

(a)

$$
\begin{aligned}
\frac{dy(t)}{dt} + 10y(t) &= \int_{-\infty}^{\infty} x(\tau)z(t-\tau)d\tau - x(t) \\
&= x(t) * z(t) - x(t) \\
&= x(t) * [z(t) - \delta(t)]
\end{aligned}
$$

Taking the FT of both sides

$$
j\omega Y(j\omega) + 10Y(j\omega) = X(j\omega)[Z(j\omega) - 1]
$$

which yields the transfer function

$$
\begin{aligned}
H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} = \frac{Z(j\omega)-1}{j\omega+10} = \frac{1/(j\omega+1)+3-1}{j\omega+10} \\
&= \frac{2j\omega+3}{(j\omega+1)(j\omega+10)}
\end{aligned}
$$

(b) The impulse response if found using partial fraction expansion and taking the inverse

FT of the transfer function.

$$H(j\omega) = \frac{2j\omega + 3}{(j\omega+1)(j\omega+10)} = \frac{A}{(j\omega+1)} + \frac{B}{(j\omega+10)}$$
$$A = [(j\omega+1)H(j\omega)]_{j\omega=-1} = \left[\frac{2j\omega+3}{j\omega+10}\right]_{j\omega=-1} = \frac{1}{9}$$
$$B = [(j\omega+10)H(j\omega)]_{j\omega=-10} = \left[\frac{2j\omega+3}{j\omega+1}\right]_{j\omega=-10} = \frac{-17}{-9} = \frac{17}{9}$$
$$H(j\omega) = \frac{1/9}{(j\omega+1)} + \frac{17/9}{(j\omega+10)}$$
$$h(t) = \frac{1}{9}e^{-t}u(t) + \frac{17}{9}e^{-10t}u(t)$$

9. (OW 4.51)

Solution

(a) By definition of an inverse system,

$$h(t) * g(t) = \delta(t)$$
$$H(j\omega)G(j\omega) = 1$$

(b) Call the signal in Figure P4.50 $y(t)$.

i. Since the the FT at $\omega = 0$ represents the DC component of the signal $y(t)$ it means

$$Y(j0) = \frac{1}{2}.$$

However, since $H(j0) = 0$ it is impossible to have $Y(j0) = H(j0)X(j0) = 0 \cdot X(j0) = 0.5$. Therefore, an $x(t)$ does not exisit to result in the output signal $y(t)$.

ii. The system is not invertible because the inverse system $G(j\omega) = 1/H(j\omega)$ is not defined for all ω values because of $H(j\omega) = 0$ regions.

(c) The spectrum of the auditorium echo acoustics are

$$h(t) = \sum_{k=0}^{\infty} e^{-kT}\delta(t - kT)$$
$$H(j\omega) = \sum_{k=0}^{\infty} e^{-kT}e^{-j\omega kT} = \sum_{k=0}^{\infty} e^{-(j\omega+1)Tk}$$
$$= \frac{1}{1 - e^{-(j\omega+1)T}} \qquad \text{using the infinite sum formula}$$

To remove the echoes, we must find the inverse system using the relationship in (a)

$$G(j\omega) = \frac{1}{H(j\omega)} = 1 - e^{-(j\omega+1)T}$$

(d)

$$H(j\omega) = 2 + \frac{1}{j\omega} + \pi\delta(\omega) = \frac{2j\omega + \pi j\omega\delta(\omega) + 1}{j\omega} = \frac{2j\omega + 1}{j\omega}$$

The inverse system can be found as

$$G(j\omega) = \frac{1}{H(j\omega)} = \frac{X(j\omega)}{Y(j\omega)} = \frac{j\omega}{2j\omega + 1}$$

The differential equation can be found by cross-multiplying and taking the inverseve FT

$$j\omega Y(j\omega) = X(j\omega)[2j\omega + 1]$$
$$\frac{d}{dt}y(t) = 2\frac{d}{dt}x(t) + x(t)$$

with $x(t)$ and $y(t)$ the input and output respectively of the given system $h(t)$.

(e) The differential equation specifies the input output relationship

$$Y(j\omega)[(j\omega)^2 + 6j\omega + 9] = X(j\omega)[(j\omega)^2 + 3j\omega + 2]$$

The transfer functions are found by cross multiplication as

$$G(j\omega) = \frac{X(j\omega)}{Y(j\omega)} = \frac{(j\omega)^2 + 6j\omega + 9}{(j\omega)^2 + 3j\omega + 2} = \frac{(j\omega + 3)^2}{(j\omega + 2)(j\omega + 1)}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{(j\omega)^2 + 3j\omega + 2}{(j\omega)^2 + 6j\omega + 9} = \frac{(j\omega + 2)(j\omega + 1)}{(j\omega + 3)^2}$$

From $G(j\omega)$ the differential equation is found by cross multiplying and taking the inverse FT

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = \frac{d^2}{dt^2}x(t) + 6\frac{d}{dt}x(t) + 9x(t)$$

Notice the inverse differential equation just exchanges $x(t)$ and $y(t)$ in our initial LTI system. The impulse responses of our two systems are found by polynomial division, partial fraction expansion, and iFT.

$$G(j\omega) = 1 + \underbrace{\frac{3j\omega + 7}{(j\omega + 2)(j\omega + 1)}}_{G_1(j\omega)} = 1 + \frac{A}{j\omega + 2} + \frac{B}{j\omega + 1}$$

$$A = [(j\omega + 2)G_1(j\omega)]_{j\omega=-2} = \left[\frac{3j\omega + 7}{j\omega + 1}\right]_{j\omega=-2} = \frac{1}{-1} = -1$$

$$B = [(j\omega + 1)G_1(j\omega)]_{j\omega=-1} = \left[\frac{3j\omega + 7}{j\omega + 2}\right]_{j\omega=-1} = \frac{4}{1} = 4$$

$$G(j\omega) = 1 + \frac{1}{j\omega + 2} + \frac{4}{j\omega + 1}$$
$$g(t) = \delta(t) - e^{-2t}u(t) + 4e^{-t}u(t)$$

$$H(j\omega) = 1 - \underbrace{\frac{3j\omega + 7}{(j\omega + 3)^2}}_{H_1(j\omega)} = 1 - \frac{A}{(j\omega+3)^2} - \frac{B}{j\omega + 3}$$
$$A = \left[(j\omega+3)^2 H_1(j\omega)\right]_{j\omega=-3} = [3j\omega + 7]_{j\omega=-3} = -2$$
$$B = \left[\frac{d}{dj\omega}(j\omega+3)^2 H_1(j\omega)\right]_{j\omega=-3} = \left[\frac{d}{dj\omega}3j\omega + 7\right]_{j\omega=-3} = 3$$
$$H(j\omega) = 1 + \frac{2}{(j\omega + 3^2} - \frac{3}{j\omega+3}$$
$$h(t) = \delta(t) + 2te^{-3t}u(t) - 3e^{-3t}u(t)$$

10. Correlation

(a) Let the correlation be defined as

$$r(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t+\tau)\, d\tau.$$

Express $R(j\omega) = \mathcal{F}\{r(t)\}$ in terms of $X(j\omega)$ and $Y(j\omega)$, the Fourier transform of $x(t)$ and $y(t)$ respectively.

(b) Suppose $x(t) = y(t) = e^{-|t|}$. Find $R(j\omega)$ using frequency domain properties and the relationship derived in (a).

extra Find $R(j\omega)$ by evaluating the convolution integral in the time domain to get $r(t)$ and then doing the FT.

Solution

(a) Correlation is convolution without the flipping

$$r(t) = x(t) * y(-t) \longleftrightarrow R(j\omega) = X(j\omega)Y(-j\omega)$$

(b) From Example 4.2 in the book

$$X(j\omega) = \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2} = Y(j\omega).$$

This result can be used with the convolution property to find

$$\begin{aligned} R(j\omega) &= X(j\omega)Y(-j\omega) \\ &= \frac{2}{1+\omega^2}\cdot\frac{2}{1+\omega^2} = \frac{4}{(1+\omega^2)^2} \end{aligned}$$

extra The flip and drag convolution technique can be used on $y(-t)$ to find

$$r(t) = \int_{-\infty}^{\infty} e^{-|\tau|}e^{-|t+\tau|}d\tau$$

For $-t < 0 \Rightarrow t > 0$

$$\begin{aligned}
r(t) &= \int_{-\infty}^{-t} e^{\tau} e^{t+\tau} d\tau + \int_{-t}^{0} e^{\tau} e^{-(t+\tau)} d\tau + \int_{0}^{\infty} e^{-\tau} e^{-(t+\tau)} d\tau \\
&= \int_{-\infty}^{-t} e^{t+2\tau} d\tau + \int_{-t}^{0} e^{-t} d\tau + \int_{0}^{\infty} e^{-t-2\tau} d\tau \\
&= \left[e^{t} \frac{1}{2} e^{2\tau} \right]_{-\infty}^{-t} + \left[e^{-t}\tau \right]_{-t}^{0} + \left[e^{-t} \frac{1}{-2} e^{-2\tau} \right]_{0}^{\infty} \\
&= \frac{e^{t}}{2} \left[e^{-2t} - 0 \right] + te^{-t} + \frac{e^{-t}}{2} [1 - 0] \\
&= \frac{1}{2} e^{-t} + te^{-t} + \frac{1}{2} e^{-t} = e^{-t} + te^{-t} \\
&= e^{-t}(1 + t)
\end{aligned}$$

For $-t > 0 \Rightarrow t < 0$

$$\begin{aligned}
r(t) &= \int_{-\infty}^{0} e^{\tau} e^{t+\tau} d\tau + \int_{0}^{-t} e^{-\tau} e^{t+\tau} d\tau + \int_{-t}^{\infty} e^{-\tau} e^{-(t+\tau)} d\tau \\
&= \int_{-\infty}^{0} e^{t+2\tau} d\tau + \int_{0}^{-t} e^{t} d\tau + \int_{-t}^{\infty} e^{-t-2\tau} d\tau \\
&= \left[\frac{e^{t}}{2} e^{2\tau} \right]_{-\infty}^{0} + \left[e^{t}\tau \right]_{0}^{-t} + \left[\frac{e^{-t}}{-2} e^{-2\tau} \right]_{-t}^{\infty} \\
&= \frac{1}{2} e^{t} - te^{t} + \frac{1}{2} e^{t} = e^{t} - te^{t} \\
&= e^{t}(1 - t)
\end{aligned}$$

Combining the two intervals results in

$$\begin{aligned}
r(t) &= e^{-t}(1 + t)u(t) + e^{t}(1 - t)u(-t) \\
&= e^{-|t|}(1 + |t|)
\end{aligned}$$

The spectrum is computed by taking the Fourier transform

$$\begin{aligned}
R(j\omega) &= \int_{-\infty}^{\infty} r(t) e^{-j\omega t} dt \\
&= \underbrace{\int_{-\infty}^{0} e^{t}(1 - t) e^{-j\omega t} dt}_{X_1(-j\omega)} + \underbrace{\int_{0}^{\infty} e^{-t}(1 + t) e^{-j\omega t} dt}_{X_1(j\omega)} \\
&= X_1(-j\omega) + X_1(j\omega)
\end{aligned}$$

Since

$$X_1(j\omega) = \mathcal{F}\{e^{-t}(1 + t)u(t)\} = \mathcal{F}\{e^{-t}1u(t) + te^{-t}u(t)\} = \frac{1}{j\omega + 1} + \frac{1}{(j\omega + 1)^2}$$

then

$$\begin{aligned} R(j\omega) = X_1(-j\omega) + X_1(j\omega) &= \frac{1}{-j\omega+1} + \frac{1}{(-j\omega+1)^2} + \frac{1}{j\omega+1} + \frac{1}{(j\omega+1)^2} \\ &= \frac{(-j\omega+1)(j\omega+1)^2 + (j\omega+1)^2 + (j\omega+1)(-j\omega+1)^2 + (-j\omega+1)^2}{(-j\omega+1)^2(j\omega+1)^2)} \\ &= \frac{(-j\omega+1)(j\omega+1)^2 + (j\omega+1)^2 + (j\omega+1)(-j\omega+1)^2 + (-j\omega+1)^2}{(j\omega)^4 - 2(j\omega)^2 + 1} \\ &= \frac{4}{\omega^4 + 2\omega^2 + 1} = \frac{4}{(\omega^2+1)^2} \end{aligned}$$

The results from (b) and (c) are the same, although, it is much more difficult to first compute the convolution then take the Fourier transform than just using the convolution property of FT.

(Protip: Matlab is very useful for multiplying polynomials using convoultion function `conv`. In this case, the polynomial independent variable is $j\omega$. For example, the denominator of $R(j\omega)$ can be computed with the command

```
>> conv(conv([-1 1], [-1 1]), conv([1 1], [1 1])).
```

This makes it easy to compute the complex numerator.)