Homework #4
Due Th. 10/03

Note:
OW Oppenheim and Wilsky
SSS Schaum’s Signals and Systems
SPR Schaum’s Probability, Random Variables, and Random Processes

Be sure to show all your work for credit.

1. (SSS 6.71)
Solution

(a) Take the FT of both sides of the difference equation
\[ Y(e^{j\omega}) \left[ 1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{j2\omega} \right] = X(e^{j\omega}) \]
\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{j2\omega}}. \]

(b) Take the inverse FT after PFE
\[ H(e^{j\omega}) = \frac{1}{(1 - \frac{1}{2} e^{-j\omega})(1 - \frac{1}{4} e^{-j\omega})} \]
\[ = \frac{A}{1 - \frac{1}{2} e^{-j\omega}} + \frac{B}{1 - \frac{1}{4} e^{-j\omega}} \]
\[ A = \left[ H(e^{j\omega})(1 - \frac{1}{2} e^{-j\omega}) \right] e^{-j\omega=2} = 2 \]
\[ B = \left[ H(e^{j\omega})(1 - \frac{1}{3} e^{-j\omega}) \right] e^{-j\omega=3} = -1 \]
\[ H(e^{j\omega}) = \frac{2}{1 - \frac{1}{2} e^{-j\omega}} + \frac{-1}{1 - \frac{1}{4} e^{-j\omega}} \]
\[ h[n] = 2 \left( \frac{1}{2} \right)^n u[n] - \left( \frac{1}{4} \right)^n u[n] \]

(c) Use the convolution property with the result from (b) above
\[ X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \]
\[ Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}) \]
\[ = \frac{2}{(1 - \frac{1}{2} e^{-j\omega})^2 + (1 - \frac{1}{2} e^{-j\omega})(1 - \frac{1}{4} e^{-j\omega})} \]
\[ = \frac{2}{(1 - \frac{1}{2} e^{-j\omega})^2 - \frac{2}{1 - \frac{1}{2} e^{-j\omega}} + \frac{1}{1 - \frac{1}{4} e^{-j\omega}}} \]
\[ y[n] = 2(n+1) \left( \frac{1}{2} \right)^2 u[n] - \left( \frac{1}{2} \right)^2 u[n] + \left( \frac{1}{4} \right)^2 u[n] \]
\[ = \left( \frac{1}{4} \right)^2 u[n] + 2n \left( \frac{1}{2} \right)^2 u[n] \]
2. (SSS 6.73)  
\textbf{Solution}  
Notice that the ideal highpass filter can be defined as a constant minus an ideal lowpass filter. (It can also be defined using a frequency shift of a LP filter).

\[
H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})  \\
H_{lp}(e^{j\omega}) = \begin{cases} 
1 & \omega \leq \omega_c \\
0 & \omega_c < |\omega| \leq \pi
\end{cases}
\]

\[
\mathcal{F}^{-1}\left\{1 - H_{lp}(e^{j\omega})\right\} = \delta[n] - \frac{\sin \omega_c n}{\pi n}.
\]

3. (OW 5.21 (a),(b),(d),(i),(k))  
\textbf{Solution}  
(a) The signal can be decomposed into a short sequence as

\[
x[n] = u[n-2] - u[n-6] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]
\]

\[
X(e^{j\omega}) = e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}
\]

(b) Use the Fourier Transform equation to find

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{n} \left(\frac{1}{2}\right)^{-n} e^{-j\omega n}
\]

\[
= \sum_{n=1}^{\infty} \left(\frac{1}{2} e^{j\omega}\right)^n e^{j\omega}
\]

\[
= \frac{e^{j\omega}}{2(1 - \frac{1}{2} e^{j\omega})}
\]

(d) Use the FT equation to find

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{0} 2^n \sin\left(\frac{\pi}{4} n\right) e^{-j\omega n}
\]

\[
= \sum_{n=0}^{\infty} 2^n \sin\left(\frac{\pi}{4} (-n)\right) e^{j\omega n}
\]

\[
= \frac{1}{2j} \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n e^{-j\pi n/4} e^{j\omega n} - \left(\frac{1}{2}\right)^n e^{j\pi n/4} e^{j\omega n}\right]
\]

\[
= \frac{1}{2j} \left[\frac{1}{2} e^{j(\omega - \pi/4)} - \frac{1}{2} e^{j(\omega + \pi/4)}\right]
\]

(i) Notice that \(x[n]\) is periodic with a period of \(N = 6\) and only defined for 6 samples.
Therefore to find the FT, first find the Fourier series coefficients.

\[ a_k = \frac{1}{6} \sum_{n=0}^{4} e^{-jk\omega_0 n} \]

\[ = \frac{1}{6} \left[ 1 - e^{-j5\pi k/3} \right] \]

Finding the Fourier transform then

\[ X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} a_k \delta \left( \omega - \frac{2\pi k}{6} \right). \]

(k) Use the multiplication property to find the FT as periodic convolution in the frequency domain.

\[ x_1[n] = \frac{\sin(\pi n/5)}{\pi n} \leftrightarrow X_1(e^{j\omega}) = \begin{cases} 1 & |\omega| < \frac{\pi}{5} \\ 0 & \frac{\pi}{5} < |\omega| \leq \pi. \end{cases} \]

\[ x_2[n] = \cos(\frac{7\pi}{2} n) = \cos(\frac{\pi}{2} n) \leftrightarrow X_2(e^{j\omega}) = \pi \{ \delta(\omega - \frac{\pi}{2}) + \delta(\omega + \frac{\pi}{2}) \} \]

in the range of \(-\pi < \omega < \pi\). (E.g. over a \(2\pi\) interval only required since convolution is only over a period).

\[ X(e^{j\omega}) = \begin{cases} 1 & \frac{3\pi}{10} < |\omega| < \frac{7\pi}{10} \\ 0 & \text{else} \end{cases} \]

in the range \(-\pi < \omega < \pi\).

4. (OW 5.22 (a),(c),(d),(e),(f))

Solution

(a) Using the inverse Fourier Transform equation

\[ x[n] = \int_{-\pi/4}^{\pi/4} e^{j\omega n} d\omega + \int_{3\pi/4}^{\pi/4} e^{j\omega n} d\omega \]

\[ = \frac{1}{\pi n} \left[ \sin(3\pi n/4) - \sin(\pi n/4) \right]. \]

(c) Define an intermediate signal

\[ X_1(e^{j\omega}) = \begin{cases} 1 & |\omega| < \pi \\ 0 & \text{else} \end{cases} \]

\[ x_1[n] = \frac{\sin(\pi n)}{\pi n}. \]

The signal \(x[n]\) is found using the time-shift property

\[ X(e^{j\omega}) = e^{j\omega/2} X_1(e^{j\omega}) \]

\[ x[n] = x_1[n - 0.5] = \frac{\sin \pi(n - 0.5)}{\pi(n - 0.5)}. \]
(d) This is solved most simply using trig identities.

\[
X(e^{j\omega}) = \frac{1 + \cos(2\omega)}{2} + \frac{1 - \cos(3\omega)}{2} \\
= \frac{1}{2} + \frac{1}{4} [e^{j2\omega} + e^{-j2\omega}] + \frac{1}{2} - \frac{1}{4} [e^{j3\omega} + e^{-j3\omega}] \\
x[n] = \delta[n] + \frac{1}{4} \delta[n + 2] + \frac{1}{4} \delta[n - 2] - \frac{1}{4} \delta[n + 3] - \frac{1}{4} \delta[n - 3].
\]

(e) Notice this is the form of the FT of a periodic signal with \( a_k = (-1)^k \) for \( N = 4 \). There
for the signal is

\[
x[n] = \sum_{k=0}^{3} (-1)^k e^{jk(\pi/2)n} \\
= 1 - e^{j\pi n/2} + e^{j\pi n} - e^{j3\pi n/2}.
\]

(f) Use PFE techniques

\[
X(e^{j\omega}) = \frac{e^{-j\omega}}{1 - \frac{1}{5} e^{-j\omega}} - \frac{1/5}{1 - \frac{1}{5} e^{-j\omega}} \\
x[n] = \left(\frac{1}{5}\right)^{(n-1)} u[n - 1] - \frac{1}{5} \left(\frac{1}{5}\right)^n u[n].
\]

5. (OW 5.28)

**Solution**

Define

\[
Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})G(e^{j(\omega-\theta)})d\theta = 1 + e^{-j\omega} \\
y[n] = x[n]g[n] = \delta[n] + \delta[n - 1]
\]

(a) Solve for \( g[n] \) in the time domain

\[
g[n]x[n] = g[n](-1)^n = \delta[n] + \delta[n - 1] \\
\Rightarrow g[n] = (-1)^{-n}(\delta[n] + \delta[n - 1]) \\
= \delta[n] + (-1)^{1-n}\delta[n - 1] \\
= \delta[n] - \delta[n - 1].
\]

This is a unique answer.

(b) This problem may be solved in parts. Since \( x[n] = 0 \) for \( n < 0 \), the product between
the two will be zero and therfore the value of \( g[n] \) is not important (the product should
have no values for \( n < 0 \)). In a manner similar to part (a), solve for \( n = 0 \) and \( n = 1 \).
Finally, since the product \( y[n] \) has no values beyond \( n = 1 \), \( g[n] \) must not have any value
in this time range. Together this results in

\[
g[n] = \begin{cases} 
1 & n = 0 \\
2 & n = 1 \\
0 & n > 1 \\
\text{any value} & n < 0 
\end{cases}
\]

Since there are arbitrary values for \( n < 0 \), this is not a unique solution.
6. (OW 5.29)

Solution

(a) Solve using partial fraction techniques. Let the system output be $y[n] = x[n] * h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$, with

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}.$$ 

(i) The output is found as

$$X(e^{j\omega}) = \frac{1}{1 - \frac{3}{4} e^{-j\omega}}.$$ 

$$Y(e^{j\omega}) = \left[ \frac{1}{1 - \frac{3}{4} e^{-j\omega}} \right] \left[ \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \right].$$

$$y[n] = \frac{3}{4} [n] - 2 \left( \frac{1}{2} \right)^{n} [n].$$

(ii) The output is found as

$$X(e^{j\omega}) = \frac{1}{(1 - \frac{1}{4} e^{-j\omega})^2}.$$ 

$$Y(e^{j\omega}) = \left[ \frac{1}{(1 - \frac{1}{4} e^{-j\omega})^2} \right] \left[ \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \right].$$

$$y[n] = -2 \left( \frac{1}{4} \right)^{n} [n] - (n + 1) \left( \frac{1}{4} \right)^{n} [n] + 4 \left( \frac{1}{2} \right)^{n} [n].$$

(iii) This problem is most easily solved by noticing that this is an eigenvalue problem $z^n \rightarrow H(z)z^n$.

$$x[n] = (-1)^n \rightarrow y[n] = H(z)|_{z=-1}(-1)^n = H(e^{j\omega})|_{\omega=\pi}(-1)^n$$

$$y[n] = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \bigg|_{\omega=\pi} (-1)^n = \frac{2}{3} (-1)^n.$$ 

(b) Proceeding as in part (a)

$$H(e^{j\omega}) = \frac{1/2}{1 - \frac{1}{2} e^{j\pi/2} e^{-j\omega}} + \frac{1/2}{1 - \frac{1}{2} e^{-j\pi/2} e^{-j\omega}}.$$ 

(i) Multiply and PFE,

$$y[n] = \frac{-j}{2(1-j)} \left( \frac{j}{2} \right)^{n} u[n] + \frac{1}{2(1+j)} \left( \frac{-j}{2} \right)^{n} u[n] + \frac{1}{2} \left( \frac{1}{2} \right)^{n} u[n].$$

(ii) Using the eigen principal again

$$y[n] = \frac{4}{3} \cos \left( \frac{\pi}{2} n \right).$$
(c) Use multiplication in frequency domain before inverse (use `conv.m` in Matlab to make this easy)

\[ Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = 3e^{j5\omega} + e^{j4\omega} - e^{j3\omega} - 3e^{j2\omega} + e^{j\omega} + 1 + 6e^{-j\omega} - 2e^{-j3\omega} + 4e^{-j5\omega} \]

\[ y[n] = 3\delta[n + 5] + \delta[n + 4] - \delta[n + 3] - 3\delta[n + 2] + \delta[n + 1] + \delta[n] + 6\delta[n - 1] - 2\delta[n - 3] + 4\delta[n - 5]. \]

7. (OW 5.33)

**Solution**

(a) Taking the transform of both sides makes it possible to solve for the frequency response.

\[ Y(e^{j\omega}) \left[ 1 + \frac{1}{2}e^{-j\omega} \right] = X(e^{j\omega}). \]

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 + \frac{1}{2}e^{-j\omega}}. \]

(b) For the following inputs:

(i) For this input

\[ X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}. \]

\[ Y(e^{j\omega}) = \left[ \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \right] \left[ \frac{1}{1 + \frac{1}{2}e^{-j\omega}} \right] = \frac{1}{2} + \frac{1}{2}\left( \frac{1}{2} \right)^n u[n] + \frac{1}{2} \left( -\frac{1}{2} \right)^n u[n]. \]

(ii) For this input

\[ X(e^{j\omega}) = \frac{1}{1 + \frac{1}{2}e^{-j\omega}}. \]

\[ Y(e^{j\omega}) = \frac{1}{(1 + \frac{1}{2}e^{-j\omega})^2}, \]

\[ y[n] = (n + 1) \left( -\frac{1}{2} \right)^n u[n]. \]

(iii) For this input

\[ X(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}. \]

\[ Y(e^{j\omega}) = 1. \]

\[ y[n] = \delta[n]. \]
(iv) For this input

\[ X(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}. \]

\[ Y(e^{j\omega}) = \left[ 1 - \frac{1}{2}e^{-j\omega} \right] \left[ \frac{1}{1 + \frac{1}{2}e^{-j\omega}} \right] \]

\[ = \frac{1}{1 + \frac{1}{2}e^{-j\omega}} + \frac{-1/2e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}. \]

\[ y[n] = \left( -\frac{1}{2} \right)^n u[n] - \frac{1}{2} \left( -\frac{1}{2} \right)^{(n-1)} u[n-1]. \]

(c) Using the same techniques as in (b):

(i) For this input

\[ Y(e^{j\omega}) = \left[ 1 - \frac{1}{4}e^{-j\omega} \right] \left[ \frac{1}{1 + \frac{1}{2}e^{-j\omega}} \right] \]

\[ = \frac{1}{\left( 1 + \frac{1}{2}e^{-j\omega} \right)^2} + \frac{-1/4e^{-j\omega}}{\left( 1 + \frac{1}{2}e^{-j\omega} \right)^2}. \]

\[ y[n] = (n + 1) \left( -\frac{1}{2} \right)^n u[n] - \frac{1}{4} n \left( -\frac{1}{2} \right)^{(n-1)} u[n-1]. \]

(ii) For this input

\[ Y(e^{j\omega}) = \left[ 1 + \frac{1}{2}e^{-j\omega} \right] \left[ \frac{1}{1 + \frac{1}{2}e^{-j\omega}} \right] \]

\[ = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}. \]

\[ y[n] = \left( \frac{1}{4} \right)^n u[n]. \]

(iii) For this input

\[ Y(e^{j\omega}) = \left[ \frac{1}{(1 - \frac{1}{4}e^{-j\omega})(1 + \frac{1}{2}e^{-j\omega})} \right] \left[ \frac{1}{1 + \frac{1}{2}e^{-j\omega}} \right] \]

\[ = \frac{2/3}{\left( 1 + \frac{1}{2}e^{-j\omega} \right)^2} + \frac{2/9}{1 + \frac{1}{2}e^{-j\omega}} + \frac{1/9}{1 - \frac{1}{4}e^{-j\omega}}. \]

\[ y[n] = \frac{2}{3} (n + 1) \left( -\frac{1}{2} \right)^n u[n] + \frac{2}{9} \left( -\frac{1}{2} \right)^n u[n] + \frac{1}{9} \left( \frac{1}{4} \right)^n u[n]. \]

(iv) For this input

\[ Y(e^{j\omega}) = \left[ 1 + 2e^{-j3\omega} \right] \left[ \frac{1}{1 + \frac{1}{2}e^{-j\omega}} \right] \]

\[ = \frac{1}{1 + \frac{1}{2}e^{-j\omega}} + \frac{2e^{-3j\omega}}{1 + \frac{1}{2}e^{-j\omega}}. \]

\[ y[n] = \left( -\frac{1}{2} \right)^n u[n] + 2 \left( -\frac{1}{2} \right)^{(n-3)} u[n - 3]. \]
8. (OW 5.48)

Solution

(a) Take the FT of each equation, solve for $W(e^{j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{3 - \frac{1}{2}e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

$$= \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{1}{1 - \frac{1}{4}e^{-j\omega}}.$$

Take the inverse transform to find

$$h[n] = 4 \left( \frac{1}{2} \right)^n u[n] - \left( \frac{1}{4} \right)^n u[n].$$

(b) Using the result above for $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$, cross multiply, and take the inverse transform to find

$$y[n] - \frac{3}{4}y[n - 1] + \frac{1}{8}y[n - 2] = 3x[n] - \frac{1}{2}x[n - 1].$$

9. (OW 5.49)

Solution

(a) Check for linearity and time-invariance.

(i) [Linear] The system is linear.

(ii) [Not TI] Define $y_2[n] = y[n - n_0]$ and $y_1[n] = f(x_1[n]) = f(x[n - n_0])$.

$$Y_2(e^{j\omega}) = e^{-jn_0}Y(e^{j\omega})$$

$$Y_1(e^{j\omega}) = 2X_1(e^{j\omega}) + e^{-j\omega}X_1(e^{j\omega}) - \frac{d}{d\omega}X_1(e^{j\omega})$$

$$= e^{-jn_0} \left[ 2X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - \frac{d}{d\omega}X(e^{j\omega}) \right] + jn_0e^{-jn_0}X(e^{j\omega})$$

$$= e^{-jn_0}Y(e^{j\omega}) + jn_0e^{-jn_0}X(e^{j\omega}) \neq Y_2(e^{j\omega}).$$

(iii) For $x[n] = \delta[n], X(e^{j\omega}) = 1$ and

$$Y(e^{j\omega}) = 2 + e^{-j\omega}.$$

Therefore, taking the inverse transform gives

$$y[n] = 2\delta[n] + \delta[n - 1].$$

(b) Note that this results in convolution with an ideal LP filter in the frequency domain

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{\omega - \pi/4}^{\omega + \pi/4} X(e^{j\theta})H(e^{j(\omega - \theta)})d\theta.$$

Therefore, this is multiplication in the time domain with a sinc,

$$y[n] = 2x[n] \frac{\sin(\pi n/4)}{n}.$$
10. (OW 5.51)

Solution

(a) Take the FT of $h[n]$, combine, cross multiply, and inverse transform to get the difference equation.

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{1/2}{1 - \frac{1}{4}e^{-j\omega}}$$

$$= \frac{3/2 - \frac{1}{2}e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

Therefore the difference equation is

$$y[n] = \frac{3}{4}y[n - 1] + \frac{1}{8}y[n - 2] = \frac{3}{2}x[n] - \frac{1}{2}x[n - 1].$$

(b) (i) Note this system is the cascade of two Direct Form II sections. In a DFII structure, the $a_k$ coefficients are on the feedback path and the $b_k$ coefficients are in the feedforward path with the form

$$1 - \sum_k a_k y[n - k] = \sum_k b_k x[n - k].$$

Therefore, the system can be written as $Y(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})X(e^{j\omega})$ with $H_1(e^{j\omega})$ defined by the difference equation

$$y_1[n] - \frac{1}{3}y_1[n - 1] = x[n] - \frac{1}{2}x[n - 1]$$

and $H_2(e^{j\omega})$ by

$$y[n] + \frac{1}{2}y[n - 1] = \frac{1}{4}y_1[n] + y_1[n - 1].$$

This results in

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$$

$$= \left(1 - \frac{1}{4}e^{-j\omega}\right) \frac{\frac{1}{4} + e^{-j\omega}}{1 - \frac{7}{8}e^{-j\omega} - \frac{1}{2}e^{-2j\omega}}$$

$$= \frac{1}{3} - \frac{7}{8}e^{-j\omega} - \frac{1}{2}e^{-2j\omega}$$

$$1 - \frac{1}{12}e^{-j\omega} - \frac{1}{12}e^{-2j\omega}$$

This can be cross-multiplied to find the full system difference equation

$$y[n] - \frac{1}{12}y[n - 1] - \frac{1}{12}y[n - 2] = \frac{1}{4}x[n] - \frac{7}{8}x[n - 1] - \frac{1}{2}x[n - 2].$$

(ii) This was solved above in (i) as

$$H(e^{j\omega}) = \frac{\frac{1}{4} - \frac{7}{8}e^{-j\omega} - \frac{1}{2}e^{-j2\omega}}{1 - \frac{1}{12}e^{-j\omega} - \frac{1}{12}e^{-j2\omega}}.$$