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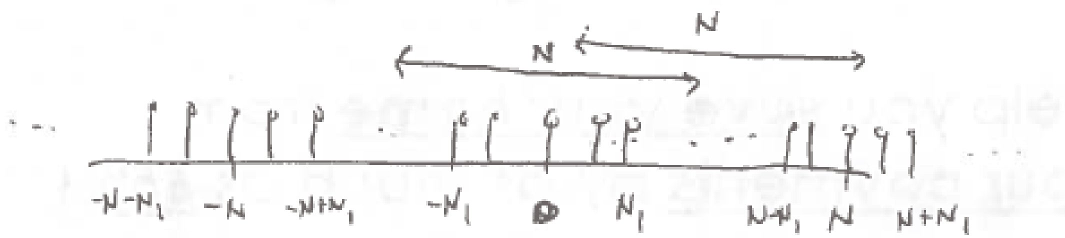
EE361: Signals and System II

Fourier Series Notes

http://www.ee.unlv.edu/~b1morris/ee361/

Periodic Rectangular Wave



- Please see book for full derivation
  - Similar to continuous case
- Must remember truncated geometric series

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

- $a_k = \begin{cases} \dfrac{2N_1+1}{N} & k = 0, \pm N, \pm 2N, \ldots \\ \dfrac{1}{N} \dfrac{\sin 2\pi k\left(N_1+\frac{1}{2}\right)/N}{\sin \pi k/N} & k \neq 0, \pm N, \pm 2N, \ldots \end{cases}$

Properties of Fourier Series

- Tables 3.1 pg 206 and Table 3.2 pg 221
  - Important to know → put on cheat sheet

- Suppose to following FS relationships
  - $x(t) \leftrightarrow a_k$ $\quad$ $x[n] \leftrightarrow a_k$
  - $y(t) \leftrightarrow b_k$ $\quad$ $y[n] \leftrightarrow b_k$

- Linearity
  - $Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k$
  - $Ax[n] + By[n] \leftrightarrow Aa_k + Bb_k$

Properties of Fourier Series II

- Time-shifting
  - $x(t - t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$
  - $x[n - n_0] \leftrightarrow a_k e^{-jk\omega_0 n_0}$
- Proof
  - Let $y(t) = x(t - t_0)$
  - $b_k = \frac{1}{T} \int_T x(t - t_0) e^{-jk\omega_0 t} dt$
    - $\tau = t - t_0$
  - $b_k = \frac{1}{T} \int_T x(\tau)\, e^{-jk\omega_0 (t + t_0)} d\tau$
  - $b_k = e^{-jk\omega_0 t_0} \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0 \tau} d\tau$
  - $b_k = e^{-jk\omega_0 t_0} a_k$

Properties of Fourier Series III

- Frequency shift
  - $e^{jM\omega_0 t}x(t) \leftrightarrow a_{k-M}$
  - $e^{jM\omega_0 n}x[n] \leftrightarrow a_{k-M}$
- Time reversal
  - $x(-t) \leftrightarrow a_{-k}$ $x[-n] \leftrightarrow a_{-k}$
  - Proof
    - Let $y(t) = x(-t)$
    - $y(t) = \sum_k a_k e^{jk\omega_0(-t)}$
      - Let $m = -k$
    - $y(t) = \sum_k a_{-m} e^{jm\omega_0 t} = \sum_k b_k e^{jk\omega_0 t}$
    - $\Rightarrow b_k = a_{-k}$

Properties of Fourier Series IV

- Periodic convolution
  - $\int_T x(\tau)y(t-\tau)d\tau \leftrightarrow Ta_k b_k$
  - $\sum_{r=<N>} x[n]y[n-r] \leftrightarrow Na_k b_k$
- Multiplication
  - $x(t)y(t) \leftrightarrow \sum_l a_l b_{k-l} = a_k * b_k$
  - $x[n]y[n] \leftrightarrow \sum_{l=<N>} a_l b_{k-l} = a_k * b_k$
    - Over a single period (since DT FS is periodic)
- Dual property
  - Convolution in the time domain = multiplication in the frequency domain
  - Multiplication in the time domain = convolution in the frequency domain

Properties of Fourier Series V

- Parseval's relation
  - $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_k |a_k|^2$
  - $\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$
  - Total avg. power in a period signal is equal to the sum average power in all its harmonics
- Time scaling
  - $x(\alpha t) \leftrightarrow a_k$ $\alpha > 0$ and periodic with period $\frac{T}{\alpha}$
  - $x_{(m)}[n] = \begin{cases} x\left[\frac{n}{m}\right] & n \text{ is multiple of } m \\ 0 & \text{else} \end{cases}$
  - $x_{(m)}[n] \leftrightarrow \frac{1}{m} a_k$ periodic with period mN
  - Not all properties are exactly the same
    - Must be careful due to constraints on periodicity of a discrete signal

Fourier Series and LTI Systems

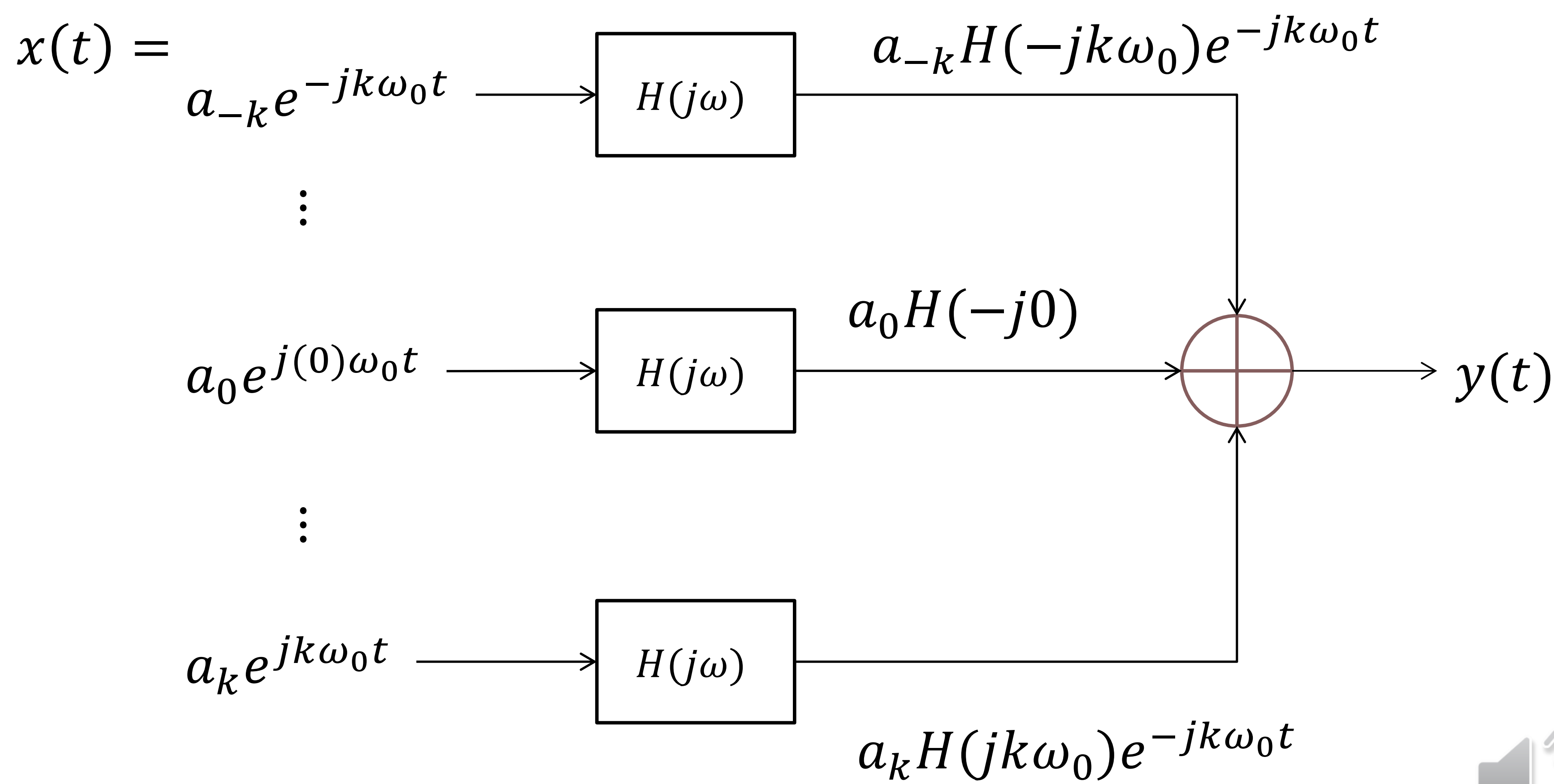
- Recall the eigen principle for LTI systems
  - $x(t) = e^{st} \leftrightarrow y(t) = H(s)e^{st}$
    - $H(s) = \int h(t)e^{-st}dt$
  - $x[n] = z^n \leftrightarrow y[n] = H(z)z^n$
    - $H(z) = \sum_k h[k]z^{-k}$
  - $s, z \in \mathbb{C}$
  - $H(s), H(z)$ are known as the system function
- Consider specific values
  - $s = j\omega, z = e^{j\omega}$
  - Results in frequency response (response to a particular frequency input e.g. sinusoid)
  - $H(j\omega) = \int h(t)e^{-j\omega t}dt$
  - $H\left(e^{j\omega}\right) = \sum_k h[n]e^{-j\omega n}$

Fourier Series and LTI Systems II

- Consider now a FS representation of a periodic signals
- $x(t) = \sum_k a_k e^{jk\omega_0 t} \rightarrow$
- $y(t) = \sum_k a_k H(jk\omega_0) e^{jk\omega_0 t}$
  - Due to superposition (LTI system)
  - Each harmonic in results in harmonic out with eigenvalue
- $y(t)$ periodic with same fundamental frequency as $x(t) \Rightarrow \omega_0$
  - $T = \frac{2\pi}{\omega_0}$ - fundamental period
- FS coefficients for $y(t)$
  - $b_k = a_k H(jk\omega_0)$
  - b_k is the FS coefficient a_k multiplied/affected by frequency response at $k\omega_0$

Fourier Series and LTI Systems III

- System block diagram



Discrete FS and LTI Systems

- $x[n] = \sum_{k=<N>} a_k e^{jk2\pi/Nn} \rightarrow$

$$y[n] = \sum_{k=<N>} a_k H(e^{j\frac{2\pi}{N}k}) e^{jk2\pi/Nn}$$

- Same idea as in the continuous case
  - Each harmonic is modified by the Frequency Response at the harmonic frequency

Example 1

- LTI system with $h[n] = \alpha^n u[n], -1 < \alpha < 1$
- Find FS of $y[n]$ given input $x[n] = \cos\frac{2\pi n}{N}$

- Find FS representation of $x[n]$
  - $\omega_0 = 2\pi/N$
  - $x[n] = \frac{1}{2}e^{j2\pi/Nn} + \frac{1}{2}e^{-j2\pi/Nn}$
  - $a_k = \begin{cases} \frac{1}{2} & k = \pm 1, \pm(N+1), \ldots \\ 0 & \text{else} \end{cases}$

- Find frequency response
  - $H(e^{j\omega}) = \sum_n h[n]e^{-j\omega n}$
  - $H(e^{j\omega}) = \sum_n \alpha^n u[n]e^{-j\omega n}$

$$H(j\omega) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$H(j\omega) = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

Let $\beta = \alpha e^{-j\omega}$

$$H(j\omega) = \frac{1}{1-\beta}$$

$$H(j\omega) = \frac{1}{1-\alpha e^{-j\omega}}$$

Example 1 II

- Use FS LTI relationship to find output
  - $y[n] = \sum_{k=<N>} a_k H(e^{jk\omega_0})\, e^{jk\omega_0 n}$
  - $y[n] = \frac{1}{2} H\left(e^{j1\frac{2\pi}{N}n}\right) e^{j1\frac{2\pi}{N}n} + \frac{1}{2} H\left(e^{-j1\frac{2\pi}{N}n}\right) e^{-j1\frac{2\pi}{N}n}$
  - $y[n] = \frac{1}{2}\left(\frac{1}{1-\alpha e^{-jk2\pi/N}}\right) e^{j\frac{2\pi}{N}n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{jk2\pi/N}}\right) e^{-j\frac{2\pi}{N}n}$

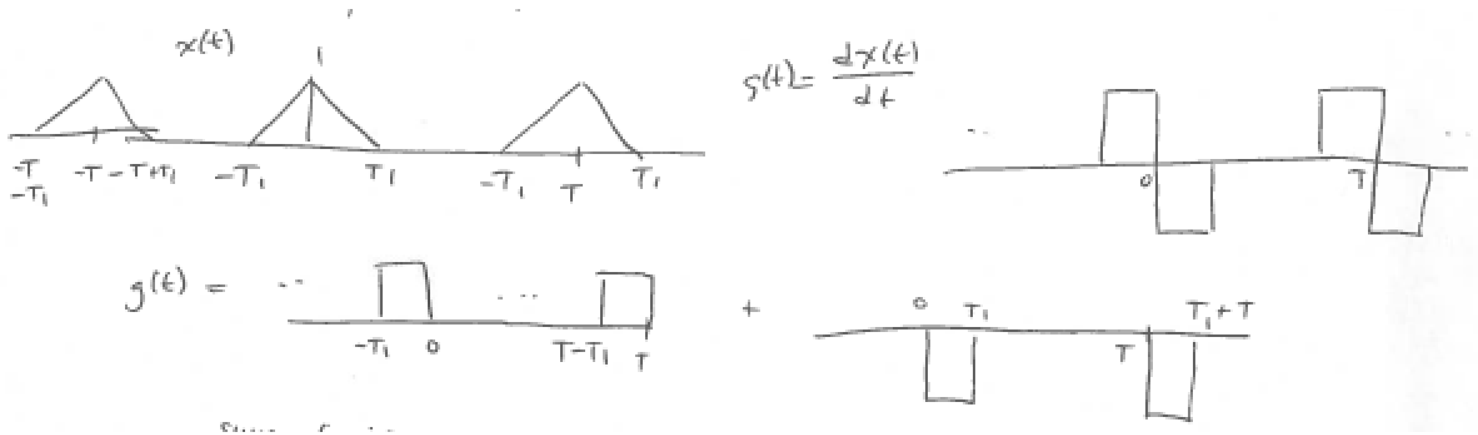
- Output FS coefficients
  - $b_k = \begin{cases} \frac{1}{2}\left(\frac{1}{1-\alpha e^{-jk2\pi/N}}\right) & k = \pm 1 \\ 0 & else \end{cases}$

Example Problem 3.7

- $x(t)$ has fundamental period T and FS a_k
- Sometimes direct calculation of a_k is difficult, at times easier to calculate transformation
  - $b_k \leftrightarrow g(t) = \frac{dx(t)}{dt}$
- Find a_k in terms of b_k and T, given $\int_T^{2T} x(t)dt = 2$
- $a_0 = \frac{1}{T}\int_T x(t)e^{-j(0)\omega_0 t}dt = \frac{1}{T}\int_T x(t)dt = 2 = \frac{2}{T}$
- From Table 3.1 pg 206
  - $b_k \leftrightarrow jk\frac{2\pi}{T}a_k \Rightarrow a_k = \frac{b_k}{jk2\pi/T}$
- $a_k = \begin{cases} 2/T & k = 0 \\ \frac{b_k}{jk2\pi/T} & k \neq 0 \end{cases}$

Example Problem 3.7 II

- Find FS of periodic sawtooth wave



- Take derivative of sawtooth
  - Results in sum of rectangular waves
- FS coefficients of rectangular waves from Table 3.2 to get $b_k \leftrightarrow g(t)$
- Then use previous result to find $a_k \leftrightarrow x(t)$
- See examples 3.6, 3.7 for similar treatment

Fourier Series Summary

- Continuous Case
- $x(t) = \sum_k a_k e^{jk\omega_0 t}$
- $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$
- Fundamental frequency ω_0
- Fundamental period $T = \frac{2\pi}{\omega_0}$

- Discrete Case
- $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$
- $a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\omega_0 n}$
- Fundamental frequency ω_0
- Fundamental period $N = \frac{2\pi}{\omega_0}$