Previously, FS allowed representation of a periodic signal as a linear combination of harmonically related exponentials

\[ x(t) = \sum_k a_k e^{jk\omega_0 t} \quad a_k = \frac{1}{T} \int_T x(t)e^{-jk\omega_0 t} dt \]

Would like to extend this (Transform Analysis) idea to aperiodic (non-periodic) signals
Intuition:
Consider a periodic signal with period $T$
Let $T \to \infty$
- Infinite period $\Rightarrow$ no longer periodic signal
Results in $\omega_0 = \frac{2\pi}{T} \to 0$
- Zero frequency space between “harmonics” $\Rightarrow$ differential $d\omega$
Envelope (like we saw with rectangle wave/sinc) defines the CTFT
CT FOURIER TRANSFORM DERIVATION II

- Will skip derivation for now
- Please see details in the book
CT FOURIER TRANSFORM PAIR

- \( x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \)  
  synthesis eq (inverse FT)

- \( X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \)  
  analysis eq (FT)

- Denote
  - \( x(t) \leftrightarrow X(j\omega) \)
  - \( X(j\omega) = \mathcal{F}\{x(t)\} \quad x(t) = \mathcal{F}^{-1}\{X(j\omega)\} \)
CTFT CONVERGENCE

- There are conditions on signal $x(t)$ for FT to exist
  
  - Finite energy (square integrable)
    
    - $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

- Dirichlet Conditions
  
  - We will not cover; see pg 290 for more discussion
CTFT FOR PERIODIC SIGNALS

CHAPTER 4.2
FT OF PERIODIC SIGNALS

- Derived FT by assuming a periodic padding of aperiodic signal $x(t)$
- What happens for FT of a periodic signal?
  - Note: periodic signal will not have finite energy
  - Cannot evaluate FT integral directly
From derivation of FT, \( X(j\omega) \) is the envelope of \( Ta_k \)
- FS coefficients are scaled samples of \( X(j\omega) \)
- Assume \( x(t) \) is periodic \([x(t) = x(t + T)]\)
- Then, \( x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \text{ with } \omega_0 = \frac{2\pi}{T} \)
- Plug into FT integral and solve
- Will not solve for now on slides \( \Rightarrow \) see book
Important property

\[ x(t) = e^{jk\omega_0 t} \leftrightarrow X(j\omega) = 2\pi \delta(\omega - k\omega_0) \]

Transform pair

\[ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \]

Each \( a_k \) coefficient gets turned into a delta at the harmonic frequency
FT OF SINUSOIDAL SIGNALS

- FT of periodic signals is important because of sinusoidal signals (cannot solve using FT integral)

- Can use insight of complex exponential ↔ shifted delta from periodic FT derivation

Important examples

\[ x(t) = \sin \omega_0 t \quad \rightarrow \quad a_1 = \frac{1}{2j} \quad \Rightarrow \quad X(j\omega) = \frac{2\pi}{-2j} \delta(\omega + \omega_0) + \frac{2\pi}{2j} \delta(\omega - \omega_0) = -\frac{\pi}{j} \delta(\omega + \omega_0) + \frac{\pi}{j} \delta(\omega - \omega_0) \]

\[ x(t) = \cos \omega_0 t \quad \rightarrow \quad a_1 = \frac{1}{2} \quad \Rightarrow \quad X(j\omega) = \pi \delta(\omega + \omega_0) + \pi \delta(\omega - \omega_0) \]
CTFT PROPERTIES AND PAIRS
CHAPTER 4.3-4.6
Properties Table 4.1 (pg 328)

- **Linearity**
  - \( x(t) \leftrightarrow X(j\omega) \)
  - \( y(t) \leftrightarrow Y(j\omega) \)
  - \( ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega) \)

- **Time shifting**
  - \( x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega) \)
  - Note, this is a phase shift of \( X(j\omega) \)

- **Conjugation**
  - \( x^*(t) \leftrightarrow X^*(-j\omega) \)
    - Remember: conjugation switches sign of imaginary part

- **Time scaling**
  - \( x(at) \leftrightarrow \frac{1}{|a|} X \left( \frac{j\omega}{a} \right) \)

- **Differentiation in time**
  - \( \frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega) \)

- **Integration in time**
  - \( \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega) \)
## Convolution/Multiplication Properties

- **Convolution**
  \[ y(t) = h(t) \ast x(t) \leftrightarrow Y(j\omega) = H(j\omega)X(j\omega) \]

- **Multiplication**
  \[ r(t) = s(t)p(t) \leftrightarrow R(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta)P(j(\omega - \theta))d\theta \]
  \[ R(j\omega) = \frac{1}{2\pi} S(j\omega) \ast P(j\omega) \]

- **Dual properties** – convolution \(\leftrightarrow\) multiplication
Be sure to bookmark this table (right next to Table 4.1 Properties)

Note in particular some very useful pairs that aren’t typical

- $t e^{-at} u(t) \leftrightarrow \frac{1}{(a+j\omega)^2}$ repeated root

- $u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$
CTFT AND LTI SYSTEMS

CHAPTER 4.7
**FIRST-ORDER EXAMPLE**

- Find impulse response $h(t)$

\[
\frac{d}{dt} y(t) + ay(t) = x(t)
\]

\[
j\omega Y(j\omega) + aY(j\omega) = X(j\omega)
\]

\[
Y(j\omega) [j\omega + a] = X(j\omega)
\]

\[
H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a + j\omega}
\]

\[
h(t) = \mathcal{F}^{-1} \left\{ \frac{1}{a + j\omega} \right\} = e^{-at} u(t)
\]
Note for $H(j\omega)$ to exist, the LTI system must have impulse response $h(t)$ that satisfies stability conditions.

FT is only for the analysis of stable LTI systems.
- For not stable systems, use Laplace Transform in Ch9.
GENERAL DIFFERENTIAL EQUATION SYSTEM

\[ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \]

- Take FT of both sides

\[ \sum_{k=0}^{N} a_k \mathcal{F} \left\{ \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^{M} b_k \mathcal{F} \left\{ \frac{d^k x(t)}{dt^k} \right\} \]

\[ \sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) \]

- Solve for frequency response

\[ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \]

- Rational form – ratio of polynomials in \( j\omega \)

- Best solved using partial fraction expansion (Appendix A)