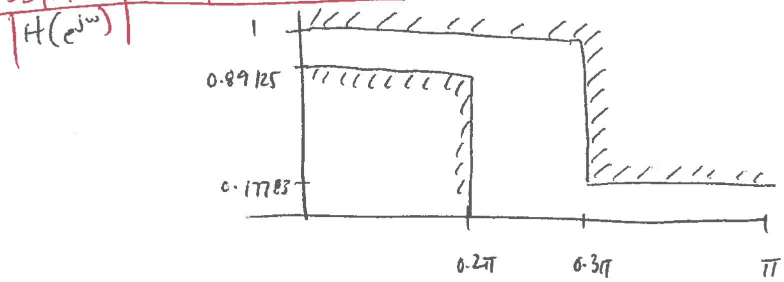
examples of design by impulse invariance

$h[n] = T_d \, h_c(nT_d)$ with relationship $\Omega = \frac{\omega}{T_d}$

example
design a low pass filter using impulse invariance — Example 7.2

① obtain DT specification



discrete specifications

$0.89125 \le |H(e^{j\omega})| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$

$|H(e^{j\omega})| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$

② find equivalent continuous specification

using $\Omega = \frac{\omega}{T_d}$

$0.89125 \le |H(j\Omega)| \le 1 \qquad 0 \le |\Omega| \le \frac{0.2\pi}{T_d}$

$|H(j\Omega)| \le 0.17783 \qquad \frac{0.3\pi}{T_d} \le |\omega| \le \frac{\pi}{T_d}$

it turns out T_d does not affect the design process

⇒ select T_d for ease (eg. $T_d = 1$ for impulse invariance)

③ Realize CT system

using a Butterworth filter this system can be realized with 6 poles evenly spaced around circle of radius 0.7032.

③ $$H_c(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

④ Convert from CT to DT system

Do partial fraction expansion on $H_c(s)$ into 1st order terms

$$H_c(s) = \sum_{k=1}^{6} \frac{A_k}{s - s_k} \xrightarrow[\text{invariance}]{\text{impulse}} H(z) = \sum_{k=1}^{6} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

example

repeat with Bilinear Transformation design — Example 7.3

② find equivalent continuous specification

must pre-warp frequencies to fit between $[-\pi, \pi]$

– again T_d parameter is not important because we go from DT ⟶ CT (would be important the other way around CT ⟶ DT) using $\Omega = \frac{2}{T_d} \tan \frac{\omega}{2}$ (select $T_d = 1$ for convenience)

$0.89125 \leq |H(e^{j\omega})| \leq 1 \qquad 0 \leq |\Omega| \leq 2 \tan\left(\frac{0.2\pi}{2}\right)$

$|H(e^{j\omega})| \leq 0.17783 \qquad 2 \tan\left(\frac{0.3\pi}{2}\right) \leq \Omega \leq \infty$

③ using a Butterworth filter (6 poles at radius 0.766)

$$H_c(s) = \frac{0.20238}{(s^2 + 0.3996s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.480s + 0.5871)}$$

④ Convert from CT to DT system

Bilinear transformation $\Rightarrow \quad s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) = \frac{0.20238}{\left(2^2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.3996\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 0.5871\right)} \cdots$$

$$= \frac{1}{4\,\frac{1-2z^{-1}+z^{-2}}{(1+z^{-1})^2} + 0.3996 \cdot 2\,\frac{1-z^{-1}}{1+z^{-1}} + 0.5871}$$

$$= \frac{(1+z^{-1})^2}{4 - 8z^{-1} + 4z^{-2} + 0.7992(1-z^{-2}) + 0.5871(1+2z^{-1}+z^{-2})}$$

$$= \frac{(1+z^{-1})^2}{\cancel{5.3863 - 8z^{-1} + 3.2006z^{-2}}\; 5.3863 - 6.8258z^{-1} + 3.7879z^{-2}} = \frac{(1+z^{-1})^2}{\cancel{1 - 1.1485z^{-1} + 0.5942z^{-2}}\; 1 - 1.2673z^{-1} + 0.7032z^{-2}}$$

Notice the 6th order zero at $z = -1$

this filter drops off rapidly

$\Rightarrow$ this corresponds to $H(j\Omega)$ @ $\Omega = \infty$

example

① Design 1st order Butterworth LP filter with 3 dB cutoff freq $\omega = \cancel{0.2\pi}\ 0.25\pi$ using bilinear transformation

$$H_a(s) = \frac{1}{1 + \left(\frac{s}{\Omega_c}\right)}$$

② transform ~~specification~~ to CT specification

$\Omega_c = \frac{2}{T_d} \tan\left(\frac{\omega_c}{2}\right) \Rightarrow \Omega_c = 2\tan\left(\frac{0.25\pi}{2}\right) = \frac{0.8284}{T_d}$

③ Realize CT system

$$H_a(s) = \frac{1}{1+\left(\frac{s}{\Omega_c}\right)} = \frac{1}{1+\left(\frac{s}{\frac{0.8284}{T_d}}\right)} = \frac{1}{1+\left(\frac{sT_d}{0.8284}\right)}$$

④ convert from CT to DT

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

notice cancellation of T_d

$$H(z) = H_a(s)\Big|_{s=} = \frac{1}{1+\left[\frac{\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)T_d}{0.8284}\right]} = \frac{1}{1+\frac{2}{0.8284}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{(1+z^{-1})}{(1+z^{-1})+2.4143\,(1-z^{-1})} = \frac{1+z^{-1}}{3.4143-1.4143\,z^{-1}} = \frac{1+z^{-1}}{1-0.4142\,z^{-1}}$$

Frequency Transformations of Lowpass IIR Filters

want to implement more than just lowpass system

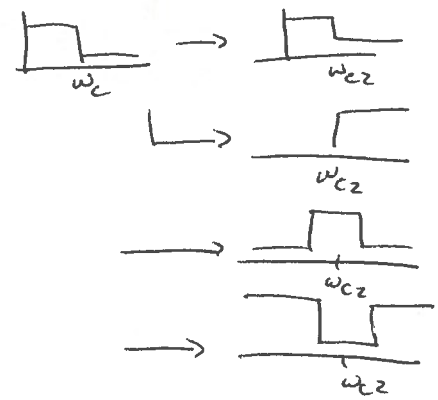
- need $\underbrace{\text{highpass, band pass,}}_{\text{cannot be transformed using impulse invariance due to aliasing effects.}}$ and band stop filters

want a transformation technique that converts lowpass prototype filter into desired ~~filter~~ frequency selective filter.

- use a transform of discrete-time lowpass filter (like we did with bilinear transform)

4 methods exist to transform a LP filter

1) LP $\longrightarrow$ LP with new cutoff frequency ω_c
2) LP $\longrightarrow$ HP with arbitrary cutoff
3) LP $\longrightarrow$ BP with arbitrary center frequency
4) LP $\longrightarrow$ BS with arbitrary center frequency



accomplished through variable substitution

$H_{lp}(Z)$ is a low pass system

$H(z)$ is the new system

need mapping from Z to z or $Z^{-1} = G(z^{-1})$ s.t.

$$H(z) = H_{lp}(Z)\Big|_{Z^{-1} = G(z^{-1})}$$

the mapping $Z^{-1} = G(z^{-1})$ must retain causality and stability

$\Rightarrow$ use Table 7.1 pg 529 for design considerations.

example of simple transformation

2) HP $Z^{-1} \Rightarrow -z^{-1}$

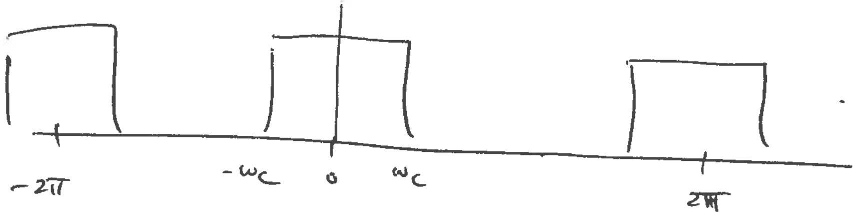
replace Z^{-1} with $-z^{-1}$ $\Rightarrow$ $H_{hp}(z) = H_{lp}(-z)$

$$H_{hp}(z) = H_{lp}(-z) = \sum_{n=-\infty}^{\infty} h_{lp}[n]\,(-z)^{-n}$$

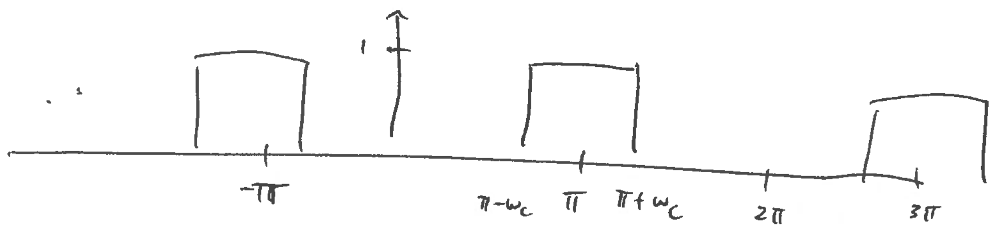
$$H_{hp}(e^{j\omega}) = H_{hp}(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h_{lp}[n]\,(-1)\,e^{-j\omega n} = \sum_{n} h_{lp}[n]\,e^{-j(\pi+\omega)n}$$

$$= H_{lp}\left(e^{j(\omega+\pi)}\right)$$

if $H_{lp}(e^{j\omega})$



$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega+\pi)})$$



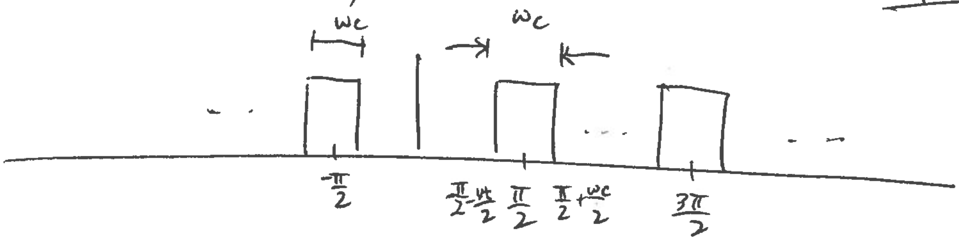
3) LP $\rightarrow$ BP $\quad z^{-1} \longrightarrow -z^{-2}$

$$H_{BP}(e^{j\omega}) = H_{LP}(-z^2)\Big|_{z=e^{+j\omega}}$$

$$= \sum_n h_{lp}[n](-z^2)^n \Big|_{z=e^{+j\omega}} = \sum_n h_{lp}[n]\, e^{j\pi n}\, e^{j\omega 2n}$$

$$= H_{lp}\left(e^{j(2\omega+\pi)}\right)$$

compress freq axis and shift.



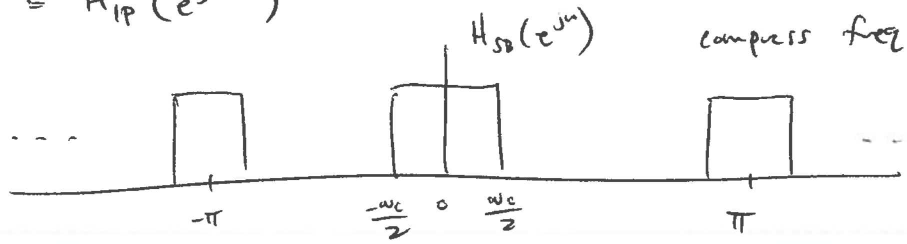
4) LP $\rightarrow$ BS. $\quad z^{-1} \longrightarrow z^{-2}$

$$H_{BS}(e^{j\omega}) = H_{lp}(z^2)\Big|_{z=e^{j\omega}}$$

$$= \sum_n h_{lp}[n](z^2)^n = \sum_n h_{lp}[n]\, e^{j\omega 2n}\Big|_{z=e^{j\omega}}$$

$$= H_{lp}(e^{j2\omega})$$

compress freq axis



In general, would need to use design formulas to generate transformations

$$z^{-1} = G(z^{-1}) \quad \text{in Table 7.1 pg 529}$$

Note these transformations preserve magnitude but not phase.

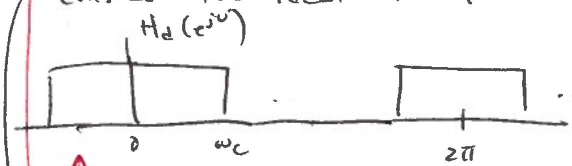
Design of FIR Filters by windowing

- IIR design based on transformations of continuous - IIR systems into discrete
- FIR design is based on direct approximation of freq / or impulse response

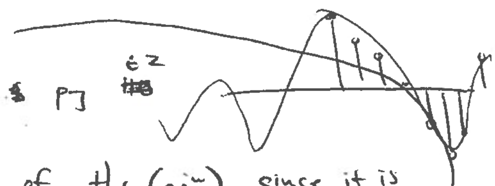
Let $H_d(e^{j\omega})$ represent the desired frequency response

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n} \qquad \text{with} \qquad h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

Consider the ideal low-pass filter



because of the sharp edges
- this requires an infinite impulse response to approximate.



$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{\sin(\omega_c n)}{\pi n}$$

Table 2.3 pg

this can be thought of as the Fourier series representation of $H_d(e^{j\omega})$ since it is periodic.

- $h_d[n]$ are the FS coefficients - indicate contribution of $e^{j\omega n}$
- build $H_d(e^{j\omega})$ from summation of periodic signals.

Since a FIR system is required, the easiest method of approximation is to truncate the coefficients of the FS, e.g. $h_d[n]$.

- less coefficients is a worse approximation.

This truncation defines a new FIR system

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{else.} \end{cases}$$

This truncation can be generalized with the use of a window function of finite duration

$$h[n] = h_d[n]\,w[n]$$

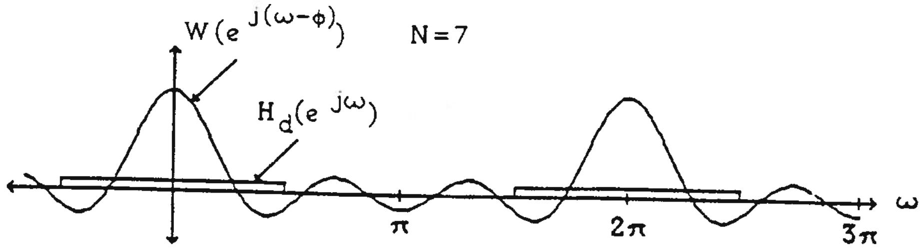
For simple truncation the window is the rectangular window

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else.} \end{cases}$$

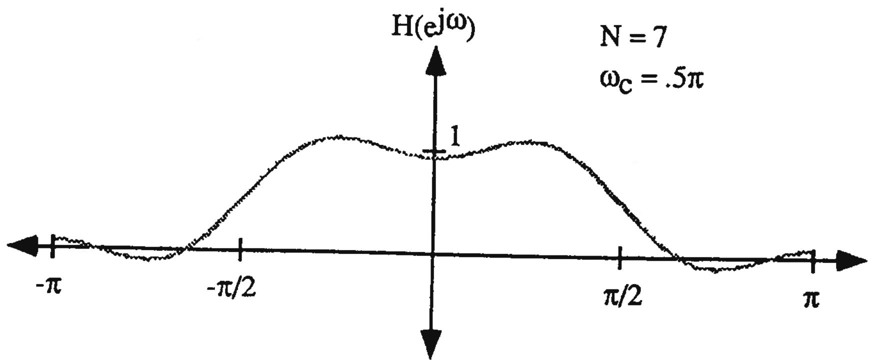
The system $H(e^{j\omega})$ can be found by FT modulation property

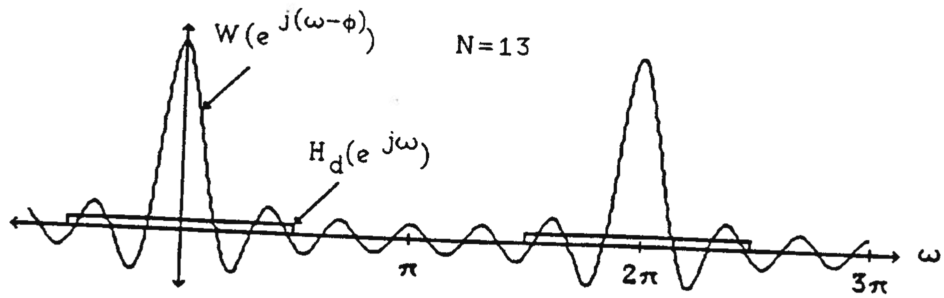
$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$

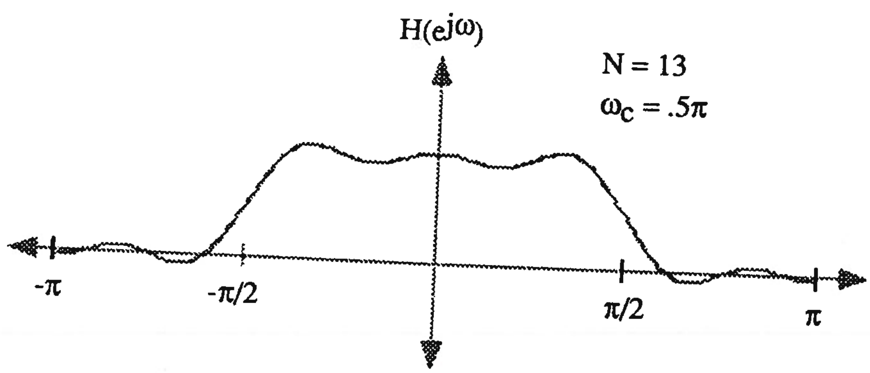
multiplication in time domain → periodic convolution in freq domain



where $W(e^{j(\omega-\phi)})$ slides across $H_d(e^{j\omega})$ from $-\pi$ to π.





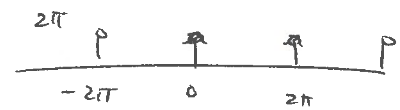


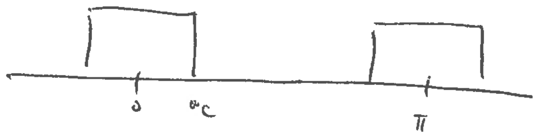
for $w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases} \longleftrightarrow W(e^{j\omega}) = \underbrace{\frac{\sin\left(\frac{\omega(M+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}}_{|W(e^{j\omega})|} e^{-j\omega\frac{M}{2}}$ $\quad \measuredangle W(e^{j\omega})$

so convolve the sinc with the ideal low pass filter $H_d(e^{j\omega})$

- causes a smearing of $H_d(e^{j\omega})$ pg 535 Fig 7.27 see handout

for $w[n] = 1 \;\forall n$ (eg. no truncation) $\longleftrightarrow W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega + 2\pi k)$





- convolution returns $2\pi \cdot H_d(e^{j\omega})\big|$

This suggests the closer the window function can get to an impulse, the better the approximation

- concentrate "mass" ~~around~~ in narrow f band around $\omega = 0$.
- make $H(e^{j\omega})$ look like $H_d(e^{j\omega})$ except at abrupt changes.

(*) However, we want $h[n]$ as short as possible for computational reasons

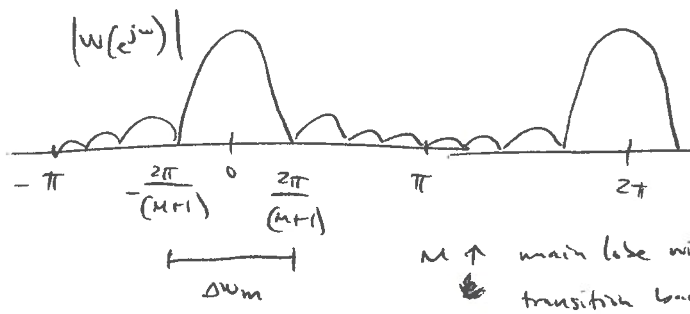
- more samples required for better approximation $\Rightarrow$ (pointier $W(e^{j\omega})$)

for the rectangular window again

pg 535 Fig 7.28

greater $M \rightarrow$ narrower main lobe delta-ish area

$\rightarrow$ ~~[illegible]~~ increased side-lobes.



$M \uparrow$ main lobe width $\downarrow$

$\uparrow$ transition band $\downarrow$

The ripple effect when approximating the ideal low-pass is ~~created~~ called the Gibbs phenomenon

- it can be softened by choosing a window with smooth transitions to zero at edges.
- reduces the height of side-lobes (thus ripples) but causes ~~longer~~ wider main lobe (thus wider transition period – less ideal cutoff frequency).

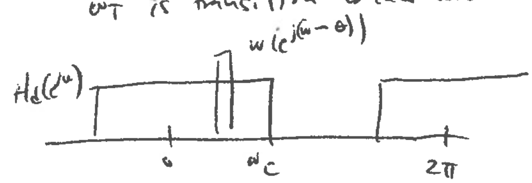
(*) Since our designs don't call for an abrupt transition we don't need a delta for "ideal" windowing

- we have a transition band – area we don't specify between passband and stop band

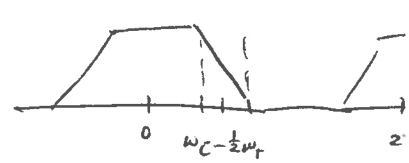
$W(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \frac{1}{2}\omega_T \\ 0 & \text{else} \end{cases}$ $\quad \omega_T$ is transition band width

FT $\updownarrow$

$w[n] = \frac{\sin(\omega_c n)}{\pi n}$



$\Rightarrow$



Commonly Used Windows

rectangular $\quad w(n) = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$

Bartlett (triangular) $\quad w(n) = \begin{cases} 2n/M & 0 \le n \le M/2,\ M \text{ even} \\ 2 - 2n/M & M/2 < n \le M \\ 0 & \text{else} \end{cases}$

Hanning $\quad w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$

Hamming $\quad w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$

Blackman $\quad w(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M}\right) + 0.08\cos\left(\frac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$

Notice all windows are symmetric about $M/2$

$$w(n) = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

this ensures that they will retain linear phase properties of $H(e^{j\omega})$

Kaiser Window Filter Design.

how should a window be selected?

- a near optimal window could be found using Bessel functions.

$$w[n] = \frac{I_0\left[\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right]}{I_0(\beta)} \quad 0 \le n \le M$$

$\alpha = M/2$

$I_0(\cdot)$ is zeroth order Bessel function of the first kind.

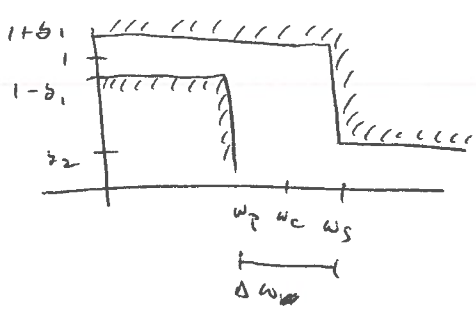
β - a shape parameter

$I_0(\cdot)$ can be approximated by power series expansion $\qquad$ pg 543 Fig 7.32

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{\left(\frac{x}{2}\right)^k}{k!}\right]^2$$

varying M and β trades-off side-lobe amplitude for main lobe width.

Given specifications of the form



define transition band $\Delta\omega = \omega_s - \omega_p$

define $\delta = \sqrt{\delta_1\delta_2}$ or $\delta = \min[\delta_1, \delta_2]$

define $A = -20\log_{10}(\delta)$

$\uparrow$
attenuation

choose β as

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases}$$

find M to satisfy A, $\Delta\omega$

$$M = \frac{A-8}{2.285\,\Delta\omega}$$

FIR Filter design using Kaiser window
(LP filter)

1) establish the filter specifications

$$\underbrace{\delta_1, \delta_2}_{\delta},\ \omega_p,\ \omega_s$$

2) determine cutoff frequency + determine $h_d[n]$

$$\omega_c = \frac{\omega_p + \omega_s}{2}$$

3) determine Kaiser window parameters

$$\left.\begin{aligned} \Delta\omega &= \omega_s - \omega_p \\ A &= -20\log_{10}(\delta) \end{aligned}\right\} \Rightarrow \begin{matrix} \beta \\ M \end{matrix}$$

4) define impulse response

$$h[n] = h_d[n]\,w[n] = \begin{cases} \dfrac{\sin \omega_c (n-\alpha)}{\pi (n-\alpha)} \cdot \dfrac{I_0\left(\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right)}{I_0(\beta)} & 0 \le n \le M \\ 0 \end{cases}$$

be sure this is causal $\alpha \neq M/2$

5) phase is linear and group delay $= M/2 = \alpha$

Example low pass filter design by Kaiser window method.

low pass filter with the following specs

$$-.0087\,dB \le 20\log_{10}|H(e^{j\omega})| \le 0.086\,dB \qquad |\omega| \le 0.4\pi$$

$$20\log_{10}|H(e^{j\omega})| \le -60\,dB \qquad |\omega| \ge 0.6\pi$$

1) establish filter specifications

$\omega_p = 0.4\pi \qquad \delta_1 = 10^{\frac{0.086}{20}} = 0.1$

$\omega_s = 0.6\pi \qquad \delta_2 = 10^{\frac{\cancel{-0.087}\ 60}{20}} = 0.001$

2) determine cutoff freq and $h_d[n]$

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.6\pi + 0.4\pi}{2} = 0.5\pi$$

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_p \\ 0 & \text{else} \end{cases} \quad \xleftrightarrow{FT} \quad h_d[n] = \frac{\sin \omega_c n}{\pi n}$$

Table 2.3
Pg 62

3) determine Kaiser window parameters

$\Delta\omega = \omega_s - \omega_p = 0.6\pi - 0.4\pi = 0.2\pi$ $\qquad \delta = \min(\delta_1, \delta_2) = 0.001$

$A = -20\log_{10}(\delta) = +60$

since $A > 50$

$\Rightarrow \beta = 0.1102(A - 8.7) = 5.653$

$$\Rightarrow M = \frac{A - 8}{2.285\,\Delta\omega} = 36.219 \xrightarrow{\text{round up}} 37$$

for ~~fixed~~ integer length, round up

4) define impulse response

$h[n] = h_d[n]\,w[n]$

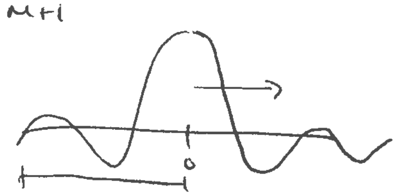
$$= \cancel{\sin \omega_c} = \frac{\sin \omega_c n}{\pi n} \cdot \frac{I_0\left[\beta\sqrt{1 - \left(\frac{n-\alpha}{\alpha}\right)^2}\right]}{I_0(\beta)} \qquad \alpha = \frac{M}{2}$$

However we know we need a causal filter of length $M+1$

$\Rightarrow$ therefore we must shift $h_d[n]$ to make it causal

$$h[n] = h_d[n - \tfrac{M}{2}]\,w[n] = h_d[n-\alpha]\,w[n]$$

$$= \frac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)} \cdot \frac{I_0\left[\beta\sqrt{1 - \left(\frac{n-\alpha}{\alpha}\right)^2}\right]}{I_0(\beta)}$$



anti-causal response