EE482: Digital Signal Processing Applications

DSP Fundamentals

http://www.ee.unlv.edu/~b1morris/ee482/
Outline

- Elementary Signals
- System Concepts
- Z-Transform
- Frequency Response
Elementary Digital Signals

- Digital signal
  - $x(n)$  $n \in \mathbb{Z}$
  - Deterministic – expressed mathematically (e.g. sinusoid)
  - Random – cannot be described exactly by equations (e.g. noise, speech)

- Unit impulse (Kronecker delta)
  - $\delta(n) = \begin{cases} 
  1, & n = 0 \\
  0, & n \neq 0 
\end{cases}$
  - Basic building block of all digital signals

- Unit step
  - $u(n) = \begin{cases} 
  1, & n \geq 0 \\
  0, & n < 0 = \sum_{k=-\infty}^{n} \delta(k) 
\end{cases}$
Sinusoidal Signals

- Continuous
  \[ x(t) = A \sin(\Omega t + \phi) = A \sin(2\pi f t + \phi) \]

- Sampled
  \[ x(n) = A \sin(\Omega nT + \phi) = A \sin(2\pi f nT + \phi) \]
    \[ \Omega = 2\pi f \]
  \[ x(n) = A \sin(\omega n + \phi) = A \sin(F\pi n + \phi) \]
    \[ \omega = \Omega T \]
Relationships Between Frequency Variables

Table 2.1

<table>
<thead>
<tr>
<th>Variable</th>
<th>Units</th>
<th>Relationships</th>
<th>Ranges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ω</td>
<td>rads/sec</td>
<td>Ω = 2πf</td>
<td>−∞ &lt; Ω &lt; ∞</td>
</tr>
<tr>
<td>f</td>
<td>cycles/sec (Hz)</td>
<td>f = Ω / 2π = ωfₛ / 2 π</td>
<td>−∞ &lt; f &lt; ∞</td>
</tr>
<tr>
<td>ω</td>
<td>rads/sample</td>
<td>ω = ΩT = 2πf / fₛ</td>
<td>−π ≤ ω ≤ π</td>
</tr>
<tr>
<td>F</td>
<td>cycles/sample</td>
<td>F = f / fₛ / 2 = ω / 2</td>
<td>−1 ≤ F ≤ 1</td>
</tr>
</tbody>
</table>

- Normalized frequency measures
  - Note: max frequency for $\pi$ or definition over a $2\pi$ interval
  - Consider $e^{j(\omega + 2\pi k)}$
Example 2.1

- \( A=2; \)
- \( f=1000; \)
- \( \text{fs} = 8000; \)
- \( n=0:31; \)
- \( w = 2\pi f/\text{fs}; \)
- \( x = A\sin(wn); \)

\[
h=\text{figure};
\%
\text{plot sampled sine}
\text{subplot}(2,1,1)
\text{plot}(n,x,'-o','linewidth',2);
xlabel('time index [n]')
ylabel('value')
\%
\text{plot analog sine}
\text{subplot}(2,1,2)
t=0:1e-5:4e-3;
\text{plot}(t,A\sin(2\pi ft),'-o','linewidth',2);
\text{hold all;}
\text{plot}(n*(1/\text{fs}),x,'-o','linewidth',2);
xlabel('time [t=n(1/f_s)]')
ylabel('value')
Block Diagram Representation

- Processing accomplished with 3 basic operations
- Addition
  - $y(n) = x_1(n) + x_2(n)$
- Multiplication
  - $y(n) = ax(n)$
- Time shift (delay)
  - $y(n) = x(n - L)$
  - Multiple delays can be implemented with a shift register (first-in, first-out buffer)(tapped delay line)

Multiplication in z domain
System Concepts

- Generic system

\[ x(n) \xrightarrow{T} y(n) \]

- Linearity
  - Additive and homogeneity (scaling) properties
  - \[ T\{ax_1(n) + bx_2(n)\} = ay_1(n) + by_2(n) \]

- Time invariance
  - Shift in input causes corresponding shift in output
  - \[ y(n - n_0) = T\{x(n - n_0)\} \]
    - To test
      - Find \( y_1(n) = y(n - n_0) \) replace \( n \) by \( n_0 \)
      - Find \( y_2(n) = T\{x(n - n_0)\} \) response of system to shifted input
LTI Systems

- Impulse response

\[ x(n) = \delta(n) \quad \rightarrow \quad LTI \quad \rightarrow \quad y(n) = h(n) \]

- Output of LTI system \( y(n) = h(n) \) to input \( x(n) = \delta(n) \)

- Convolution
  - Input-output relationship of LTI system

\[ y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n - k) = \sum_{k=-\infty}^{\infty} h(k)x(n - k) \]
General Difference Equation Systems

\[ y(n) = \sum_{k=0}^{L-1} b_k x(n - k) - \sum_{k=0}^{M} a_k y(n - k) \]

- Infinite impulse response (IIR)
  - \( h(n) \) non-zero as \( n \to \infty \)
- Finite impulse response (FIR)
  - \( h(n) \) defined over finite set of \( n \)
  - Special case of above with \( a_k = 0 \)
  - This system only has zeroes and poles at \( z = 0 \)

- Causality
  - Output only depends on previous input
  - \( h(n) = 0, \quad n < 0 \)
Z-Transform

- Very useful computational tool for studying digital systems
- Definition

\[ X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \]

- Has associated region of convergence (ROC)
  - Values of \( z \) where summation converges
- Useful summation formulas

\[ \sum_{k=0}^{N} \alpha^n = \frac{1 - \alpha^{N+1}}{1 - \alpha} \]
\[ \sum_{k=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha} \quad |\alpha| < 1 \]
Z-Transform Properties

- **Linearity**
  - \( \mathcal{Z}\{ax_1(n) + bx_2(n)\} = aX_1(z) + bX_2(z) \)

- **Time shift**
  - \( \mathcal{Z}\{x(n - k)\} = z^{-k}X(z) \)

- **Convolution**
  - \( x(n) = x_1(n) * x_2(n) \rightarrow X(z) = X_1(z)X_2(z) \)
    - **ROC** = \( R_{x1} \cap R_{x2} \)
Transfer Functions

\[ Y(z) = X(z)H(z) \]

\[ H(z) = \frac{Y(z)}{X(z)} \]

- General polynomial form from difference equation

\[
H(z) = \frac{\sum_{k=0}^{L-1} b_k z^{-k}}{1 + \sum_{k=1}^{M} a_k z^{-k}}
\]
Poles and Zeros

\[ H(z) = b_0 \frac{\prod_{k=1}^{L-1} (z - z_k)}{\prod_{k=1}^{M} (z - p_k)} = b_0 \frac{(z - z_1)(z - z_2) \ldots}{(z - p_1)(z - p_2) \ldots} \]

- **Zeros**
  - Roots of the numerator polynomial
  - Locations in z-plane that make output zero
- **Poles**
  - Roots of the denominator polynomial
  - Locations in z-plane that make output infinity (unstable)
    - System is considered unstable if the ROC doesn’t contain the unit circle (no DTFT exists)
    - Causal system → poles should be inside unit circle
Example 2.10

• \( H(z) = \frac{1}{L} \left[ \frac{1-z^{-L}}{1-z^{-1}} \right] \)
  ▫ Notice this is a polynomial in \( z^{-1} \)

• Convert to polynomial in \( z \) to get all poles and zeros

• \( H(z) = \frac{1}{L} \left[ \frac{z^{L-1}}{z^{L-z-L-1}} \right] = \frac{1}{L} \left[ \frac{z^{L-1}}{z^{L-1}(z-1)} \right] \)
  ▫ Poles
    • \((z - 1) = 0 \rightarrow z = 1\)
    • \(z^{L-1} = 0 \rightarrow \text{L-1 poles at } z = 0\)
  ▫ Zeros
    • \(z^L - 1 = 0 \rightarrow z_l = e^{j \frac{2\pi}{L} l}\)
    • \(L\) zeros even spaced around unit circle

• Matlab
  • \texttt{fvtool([1 0 0 0 0 0 0 0 1], [1 -1]);}

![Figure 2.12 Pole-zero diagram of the moving-averaging filter, \( L = 8 \)](image)
Frequency Response

• Discrete-time Fourier transform (DTFT)

\[ H(\omega) = H(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \]

- Evaluate transfer function along the unit circle \(|z| = |e^{j\omega}|\)

\[ H(\omega) = |H(\omega)|e^{\angle H(\omega)} \]

\[ |H(\omega)| = \sqrt{H(\omega)H^*(\omega)} \quad \angle H(\omega) = \arctan\left( \frac{\text{Im} H(\omega)}{\text{Re} H(\omega)} \right) \]

• Frequency response is periodic on \(2\pi\) interval and symmetric
  - Only \([0, \pi]\) interval is required for evaluation
Graphical DTFT Interpretation

- **Poles**
  - $|H(\omega)|$ gets larger closer to $\theta$

- **Zeros**
  - $|H(\omega)|$ gets smaller closer to $\theta$

- What does a highpass filter look like?

- What does a lowpass filter look like?
Discrete Fourier Transform

• Notice the DTFT is a continuous function of $\omega$
  ▫ Requires an infinite number of samples to compute (infinite sum)

• DFT is a sampled version of the DTFT
  ▫ Samples are taken at $N$ equally spaced frequencies along unit circle
    • $\omega_k = \frac{2\pi k}{N}, k = 0,1, \ldots, N-1$

$$X(k) = X(\omega)\big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi k}{N}n}$$

▫ $n$ – time index
▫ $k$ – frequency index
DFT

\[ X(k) = X(\omega) \bigg|_{\omega = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi k}{N}n} \]

- DFT can be computed very efficiently with the fast Fourier transform (FFT)
- Frequency resolution of DFT
  - \( \Delta \omega = \frac{2\pi}{N}, \quad \Delta f = \frac{f_s}{N} \)
- Analog frequency mapping
  - \( f_k = k\Delta f = \frac{kf_s}{N}, \quad k = 0, 1, ..., N - 1 \)
  - Nyquist frequency \( \frac{f_s}{2} \) corresponds to \( k = \frac{N}{2} \)
Example 2.16

- \( N = 100; \)
- \( A = 1; \)
- \( f=1000; \)
- \( fs = 10000; \)
- \( n=0:N-1; \)
- \( w = 2\pi f/fs; \)
- \( x = \sin(w*n); \)
- \( X = \text{fft}(x); \)
- \( K = \text{length}(X); \)

- \( h=\text{figure}; \)
- \( \text{subplot}(2,1,1) \)
- \( \text{plot}(0:K-1, 20*\log10(\text{abs}(X)), \quad \text{’linewidth’, 2}); \)
- \( \text{xlabel(’freq index [k]’);} \)
- \( \text{ylabel(’magnitude [dB]’);} \)
- \( \text{subplot}(2,1,2) \)
- \( \%\text{convert index to freq} \)
- \( f = (0:K-1) * fs/N; \)
- \( \text{plot}(f, 20*\log10(\text{abs}(X)), \quad \text{’linewidth’, 2}); \)
- \( \text{xlabel(’freq [Hz]’);} \)
- \( \text{ylabel(’magnitude [dB]’);} \)