Reliability and fault tolerance of coverage models for sensor networks[1]

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Abstract: We study the coverage problem for sensor networks from the fault tolerance and reliability point of view. Fault tolerance is a critical issue for sensors deployed in places where they are not easily replaceable, repairable and rechargeable. Failure of one node should not incapacitate the entire network. We propose three 1-fault tolerant topologies, namely square, hexagonal and improved 8-node. We show how to extend these to *k*-fault tolerant schemes and calculate reliabilities using Markov models. The proposed models are compared to one another, as well as with the minimal coverage model of Zhang and Hou. The minimum coverage model is the most unreliable among the models, whereas the improved 8-node model is the most reliable except at the very beginning of the system, where the square model is more reliable. To our knowledge, this is the first paper which studies a pattern from the perspective of reliability.

Keywords: fault tolerance; Markov model; reliability; sensor network.

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1 Introduction

If all sensors deployed within a small area are active simultaneously, an excessive amount of energy is used, redundant data is generated, and packet collision can occur. At the same time, if areas are not covered, events can occur without being observed. A *density control function* is required to ensure that a subset of nodes is active in such a way that *coverage* and *connectivity* are maintained. *Coverage* refers to the total area currently monitored by active sensors; this needs to include the area required to be covered by the sensor networks. *Connectivity* refers to the connectivity of the sensor network modelled as a graph.

The nodes in a wireless environment are greatly dependent on the battery life and power. Minimising the energy usage of the network while keeping its functionality is a major objective in designing a reliable network. But sensors are prone to failures and disconnection. Only minimal coverage of a given region without redundancy would make such a network unattractive from a practical point of view. Therefore, it is necessary not only to design for minimal coverage but also fault tolerance features must be viewed in light of the additional sensors and energy used.

Sensors coupled with integrated circuits, known as *smart sensors*, provide high sensing from their relationship with each other and with higher level processing layers. A smart sensor is specifically designed for the targeted application as proposed by Moini (1997). Smart sensors find their applications in a wide variety of fields such as military, civilian, biomedical as well as control systems, etc. In military applications, sensors can track troop movements and help decide deployment of troops. In civilian applications, sensors can typically be applied to detect pollution, burglary, fire hazards and the like. Wireless body sensors (called *biosensors*) implanted in the body must be energy efficient, robust, lightweight and fault tolerant, as they are not easily replaceable, repairable and rechargeable. Biosensors need a dynamic, fault-tolerant network with a high reliability.

1.1 Related work

Finding a minimum number of sensors needed to cover a certain area and thus finding patterns for wireless sensor networks that are optimal in the sense of coverage is a hard problem and few results are available in the literature. For the case that both coverage and connectivity are desired for the network, Zhang and Hou (2004) have shown that a regular triangular lattice pattern is optimal when the ratio between the communication range and the sensing range, denoted as r_{cs}, is not greater than $\sqrt{3}$. This is also known as the ‘uniform sensing range model’ – nodes are placed to form equilateral triangles. This result can be also found in the work of Kershner (1939). Zhang and Hou (2004) also show that if the ratio $r_{cs} \geq 2$, then coverage implies connectivity.

For certain fixed values of connectivity, several patterns for coverage have been considered. When a network is sought to have connectivity of 1, then the strip pattern proposed by Wang et al. (2005) was shown to be asymptotically optimal for any ratio r_{cs} (Bai et al., 2006). For connectivity of 2, a variant of this strip pattern was shown to be asymptotically optimal (see also Bai et al., 2006). For such a strip pattern, consider two sensors *a* and *b*, which are relatively close, i.e. the distance between them is larger than the communication range but smaller than twice the communication range. Now if *a* and *b* belong to two consecutive strips, say Strips 1 and 2, they can have a long communication path, which may be as large as the number of sensors on a strip. Sensors *a* and *b* communicate using sensors that are at the left or right end of Strips 1 and 2. Sensor *a* sends the packet towards one end (say, the right end) of Strip 1, which is forwarded by each sensor on Strip 1 in that direction. The sensor placed at the end of Strip 1, then sends the packet to another sensor placed between the ends of the two strips. This sensor in turns sends it to end of Strip 2, and then finally it is forwarded from right to left towards sensor *b*. To shorten such a communication path and to achieve four-connectivity, in a very recent paper, Bai et al. (2008) use Voronoi polygons and propose a Diamond pattern which is asymptotically optimal when the ratio $r_{cs} > \sqrt{2}$. If the ratio $r_{cs} \geq \sqrt{2}$, the Diamond pattern is the uniform sensing range model proposed by Zhang and Hou (2004). If the ratio $r_{cs} < \sqrt{2}$, the Diamond pattern is our proposed square model.

Huang and Tseng (2003) have focused on determining whether each point of a target area is covered by at least *k* sensors. Zhou et al. (2004) have extended the problem further and have considered the selection of a minimum size set of sensor nodes which are connected in such a way that each point inside the target area is covered by at least *k* sensors. Starting from the uniform sensing range model proposed by Zhang and Hou (2004), Wu and Yang (2004) have proposed two models which use sensors with different sensing ranges. Variable sensing ranges are novel and in some cases the proposed models achieve lower energy when energy consumed by the sensor is proportional to the quadratic power of its transmission range. However, both models are worse than the model of Zhang and Hou (2004) in terms of achieving better coverage: the ratios of the two proposed models between the area covered and the sum of areas covered by each sensor – in this paper we call this ratio the *efficiency* – are lower than the ratio for the uniform sensing range of Zhang and Hou (2004). Also, the second model proposed by Wu and Yang (2004) requires (for some sensors) the communication range to be almost four times larger than the sensing range, otherwise connectivity is not achieved. More precisely, the ratio of the communication range and the coverage range has to be at least $\frac{2}{\sqrt{2-\sqrt{3}}} \simeq 3.86$.

Delaunay triangulations have been used extensively for wireless networks, especially for topology control of ad hoc networks (Hu, 1993). As well, Voronoi polygons have been used in ad hoc networks for routing (Stojmenovic, 1999; Stojmenovic et al., 2003; Stojmenovic et al., 2006), boundary coverage, and distributed hashing (Carbunaru et al., 2004).

For our proposed models, the Voronoi polygons have the same topology as the placement of nodes in the corresponding model. Namely, the Voronoi polygons for the square model are squares of the same dimensions as the square model. The Voronoi polygons for the hexagonal model are hexagons of the same dimensions as the hexagonal model. The Voronoi polygons for the improved 8-node model are equilateral triangles as the uniform sensing range model proposed by Zhang and Hou (2004).

We also mention that a relay node, also called *gateway* (Gupta and Younis, 2003) or *application* node (Pan et al., 2003), acts as cluster-head in the corresponding cluster. Hao et al. (2004) propose a fault-tolerant relay node placement scheme for wireless sensor networks, and a polynomial time approximation algorithm is presented to select a set of active nodes, given the set of all the nodes.

Placing the sensors in a network that provides connectivity, coverage and fault tolerance is of interest in biomedicine. Schwiebert et al. (2001) describe how to build a theoretical artificial retina made up of smart sensors, used for reception and transmission in a feedback system. The sensors produce electrical signals that are subsequently converted by the underlying tissue into chemical signals to be sent to the brain. The fault tolerance aspect of such a network ensures that in case a sensor fails, there is no immediate need in replacing the sensor or the entire network for the artificial retina to continue its functionality.

1.2 Contribution

Fault tolerance is a critical issue depending on where the sensors are employed. The failure of one node should not incapacitate the entire network, and despite the presence of limited number of failed sensors, the system should continue to function.

We study the coverage problem from the fault tolerance point of view. In this paper, we propose three sensor array placement schemes of four, six and eight nodes, which are 1-fault tolerant. These are standard tessellations, but we note that fault tolerance and reliability of such models have never been considered. The placement of the sensors forms regular shapes. Starting from the uniform coverage model of Zhang and Hou (2004) (which we call the *minimal coverage* model), in which the placement of the sensors is a equilateral triangle, we explore other regular shapes as the square (which we call the *square* model), the hexagon (which we call the *hexagonal* model) and the hexagonal model with the centre (which we call the *improved* 8-node model), in such a way that the resulting network obtained by placing nodes in these shapes is 1-fault tolerant. Regular placement of the nodes provides a better coverage than irregular placement (Zhang and Hou, 2004). Furthermore, regular placement of the nodes together with a uniform transmission range of all sensors assures connectivities of the network: If x is within transmission range of y, then y is within transmission range of x. The improved 8-node model has one sensor whose range is larger than the other seven, but the network is still biconnected. The reason for designing a biconnected wireless network is that it has more reliable communication than a 1-connected one.

While the minimal coverage model has the best efficiency, our models are 1-fault tolerant and their efficiency is better than that of the models proposed by Zhang and Hou (2004). A 2-fault tolerant placement scheme can be obtained by overlapping two 1-fault tolerant schemes, either of the same type or a different type. Thus a k-fault tolerant scheme is obtained by overlapping k individual 1-fault tolerant models, or $k/2$ individual 2-fault tolerant models and so forth.

We show that if p is the probability of a node to fail, and assuming that p is constant over time, then the minimum coverage model has the lowest probability to function. For various values of p in the range of 0–1, either the square model or the improved 8-node model is the best.

An important aspect of fault tolerance is reliability. We develop Markov models for the four models (the minimum coverage model and our three proposed models). For each of them, we define the reliability function over time assuming a constant failure rate of the sensors. We show that the improved 8-node model is the most reliable over time; only at the very beginning the square model is more reliable. We also show that the most unreliable model among the three 1-fault tolerant models is the hexagonal model, whereas the uniform coverage model is the most unreliable.

1.3 Outline of the paper

Section 2 presents the proposed three models. In Section 3, we compare them among one another, and with the minimal coverage model from Zhang and Hou (2004). We finish with concluding remarks in Section 4. The Appendix contains detailed calculations.

2 Fault-tolerant models

Two parameters are important for a sensor node: the wireless communication range of a sensor r_C, and the sensing range r_S. They generally differ in values, and a common assumption is that $r_C \geq r_S$. Obviously, two nodes u and v, whose wireless communication ranges are r_{C_u} and r_{C_v}, respectively, can communicate directly if $dist(u,v) \leq min(r_{C_u}, r_{C_v})$. Zhang and Hou (2004) proved that if all the active sensor nodes have the same radio range r_C and the same sensing range r_S, and the radio range is at least twice of the sensing range $r_C \geq 2 \times r_S$, complete coverage of an area implies connectivity among the nodes. Under this assumption, the connectivity problem reduces to the coverage problem.

There is a trade-off between minimal coverage and fault tolerance. For the same set of sensors, a fault-tolerant model will have a smaller area to cover. Or, given an area to be covered, more sensors will be required, or the same number of sensors but with a higher values for the parameters.

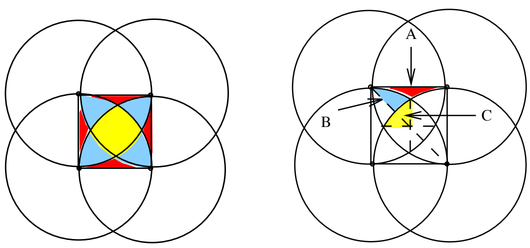
A model is k-fault tolerant if by removal of any k nodes, the network preserves its functionality. A k-fault tolerant model for the coverage problem will be able to withstand k removals: by removing any k nodes, the covered region remains the same. A 0-tolerant model will not work in case of removal of some node. To transform a 0-fault tolerant model into a 1-fault tolerant model, a straightforward approach is to either double the number of sensors at each point, or to double the sensor parameters for some sensors. Similar actions can be taken to transform a 1-fault tolerant model into a 2-fault tolerant and so forth.

In order for a *k*-fault tolerant model to be worthwhile, it has to be better than the straightforward approach. We propose three 1-fault tolerant models: a square, hexagonal and improved 8-node model (that is obtained by adding one more larger range sensor in the middle of a 7-node minimal coverage model). We assume the sensing range of the sensors in square, hexagonal and partially in the improved 8-node model to be *r*. In the improved 8-node model, there is one sensor whose sensing range is $r\sqrt{3}$. Based on the topology of these models, the assumption that the communication range is greater than twice the sensing range guarantees the connectivity of the network.

2.1 *Square fault-tolerant model*

The basic structure for the first model is composed of four sensors arranged in a square-like shape of side *r* and is shown in Figure 1(a). A 2D region of dimension $(rN) \times (rM)$, with N and M strictly positive integers, requires $(N + 1) \times (M + 1)$ sensors arranged in this pattern.

Figure 1 Square fault-tolerant model (see online version for colours)



(a) Square model (b) Areas *A*, *B* and *C*

We partitioned the square surface $S_4 = r^2$ into three areas: one area covered by exactly two sensors $\left(S_{2s}^{square}\right)$, one area covered by exactly three sensors $\left(S_{3s}^{square}\right)$ and one area covered by exactly four sensors $\left(S_{4s}^{square}\right)$. To analyse the fault tolerance of such placement, we compute the surface of each such areas (see Section A.1 for more details). Let *A*, *B* and *C* to be the disjoint areas as drawn in Figure 1(b). We observe that $\left(S_{2s}^{square}\right) = 4S_A, S_{3s}^{square} = 8S_B, S_{4s}^{square} = 4S_C$, and we obtain that:

$$\begin{cases} S_{2s}^{square} = 4r^2 - r^2\sqrt{3} - \dfrac{2\Pi r^2}{3} \\ S_{3s}^{square} = -4r^2 + 2r^2\sqrt{3} + \dfrac{\Pi r^2}{3}. \\ S_{4s}^{square} = r^2 - r^2\sqrt{3} + \dfrac{\Pi r^2}{3} \end{cases}$$

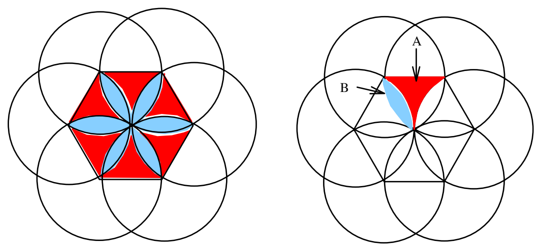
Therefore, given a 2D region of dimension $(rN) \times (rM)$, the ratio between the sensor area used and the area covered is $\dfrac{(N+1)(M+1)\Pi r^2}{NMr^2} = \dfrac{(N+1)(M+1)\Pi}{NM}$.

The area covered by two sensors is $NMS_{2s}^{square} = NM\left(4r^2 - r^2\sqrt{3} - \dfrac{2\Pi r^2}{3}\right)$. The area covered by three sensors is $NMS_{3s}^{square} = NM\left(-4r^2 + 2r^2\sqrt{3} + \dfrac{\Pi r^2}{3}\right)$. The area covered by four sensors is $NMS_{4s}^{square} = NM\left(r^2 - r^2\sqrt{3} + \dfrac{\Pi r^2}{3}\right)$.

2.2 *Hexagonal fault-tolerant model*

The basic structure for the second model is composed of six sensors arranged in a regular hexagon-like shape of side *r* and is shown in Figure 2(a). A 2D region of dimension $(rN) \times (rM)$, with N and M strictly positive integers, requires $2 + 4\left\lfloor \dfrac{N}{2} \right\rfloor \left\lfloor \dfrac{4M}{\sqrt{3}} \right\rfloor$ sensors arranged in this pattern.

Figure 2 Hexagonal fault-tolerant model (see online version for colours)



(a) Hexagonal model (b) Selected areas *A* and *B*

The hexagonal surface $S_6 = \dfrac{3\sqrt{3}}{2}$ is partitioned into two areas: one area covered by exactly two sensors $\left(S_{2s}^{hexagon}\right)$ and one area covered by exactly three sensors $\left(S_{3s}^{hexagon}\right)$. To analyse the fault tolerance of such placement, we compute the surface of each such areas (see Section A.2 for more details). Let *A* and *B* be the disjoint areas as drawn in Figure 2(*b*). We observe that $S_{2s}^{hexagon} = 6S_A$ and $S_{3s}^{hexagon} = 6S_B$, and we obtain that:

$$\begin{cases} S_{2s}^{hexagon} = \dfrac{9r^2\sqrt{3}}{2} - 2\Pi r^2 \\ S_{3s}^{hexagon} = -3r^2\sqrt{3} + 2\Pi r^2 \end{cases}.$$

Given a 2D region of dimension $(rN) \times (rM)$, the ratio between the sensor area used and the area covered is as follows:

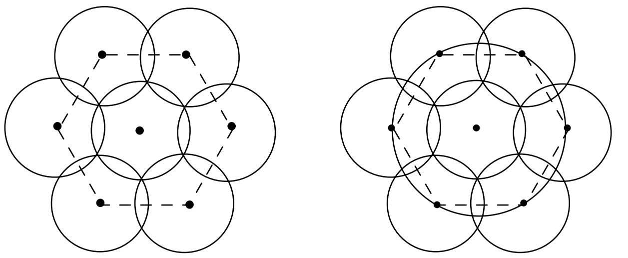
$$\frac{\left(2 + 4\left\lfloor \frac{N}{2} \right\rfloor \left\lfloor \frac{4M}{\sqrt{3}} \right\rfloor\right)\Pi r^2}{NMr^2} = \frac{\Pi\left(2 + 4\left\lfloor \frac{N}{2} \right\rfloor \left\lfloor \frac{4M}{\sqrt{3}} \right\rfloor\right)}{NM}.$$

2.3 Improved 8-node model

Let the *minimal coverage model* be the model obtained by overlapping three 3-node models (Figure 3a). The minimal coverage model can be made 1-fault tolerant by adding a sensor whose sensing range is increased from r to $r\sqrt{3}$; we obtain the *improved 8-node model* (Figure 3b). A 2D region of dimension (rN) × (rM), with N and M strictly positive integers, requires $2+6\left\lfloor \frac{N}{2} \right\rfloor \left\lfloor \frac{4M}{\sqrt{3}} \right\rfloor$ sensors arranged in the improved 8-node model pattern.

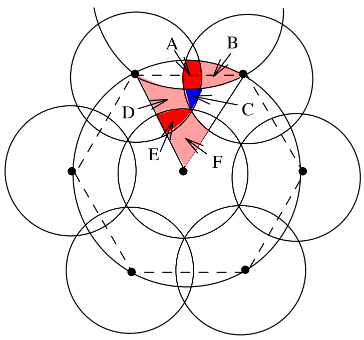
The hexagonal surface $S_8 = \frac{3\sqrt{3}}{2}$ is partitioned into three areas: one area covered by exactly two sensors $\left(S_{2s}^{improved}\right)$, one area covered by exactly three sensors $\left(S_{3s}^{improved}\right)$ and one area covered by exactly four sensors $\left(S_{4s}^{improved}\right)$ (Figure 4).

Figure 3 Improved 8-node vs. minimal coverage model



(a) Minimal coverage model (b) Improved 8-node model

Figure 4 Partitioning the improved 8-node model (see online version for colours)



To analyse the fault tolerance of such placement, we compute the surface of each such areas (see Section A.3 for more details). Let A, B, C, D, E and F be some disjoint areas as shown in Figure 4. We observe that $S_{2s}^{improved} = S_F + 2S_D$, $S_{3s}^{improved} = 2S_E + S_C + 2S_B$ and $S_{4s}^{improved} = S_A$, and we obtain the following:

$$\begin{cases} S_{2s}^{improved} = \dfrac{9r^2\sqrt{3}}{2} = 2\Pi r^2 \\ S_{3s}^{impvoved} = 2\Pi r^2 - 3r^2\sqrt{3} \\ S_{4s}^{improved} = \dfrac{\Pi r^2}{2} - \dfrac{3r^2\sqrt{3}}{4} \end{cases}.$$

Given a 2D region of dimension (rN) × (rM), the ratio between the sensor area used and the area covered is as below:

$$\frac{\left(2+6\left\lfloor \frac{N}{2} \right\rfloor \left\lfloor \frac{4M}{\sqrt{3}} \right\rfloor\right)\Pi r^2}{NMr^2} = \frac{\Pi\left(2+6\left\lfloor \frac{N}{2} \right\rfloor \left\lfloor \frac{4M}{\sqrt{3}} \right\rfloor\right)}{NM}.$$

3 Comparative results

We compare the minimum coverage model of Zhang and Hou (2004) with our proposed models in terms of the model efficiency (the area covered vs. the portion of the sensors area used for coverage), the probability for the model to function and reliability. Using a Markov model, we compute the reliability function for each pattern.

Table 1 contains comparative results. We use the following notations. The term *area* denotes the area covered by the polygonal line formed by the sensors. The term *used* denotes the portion of the sensor areas used for covering that area. (This value aids in calculating the energy used for covering the region.) This value is obtained by adding for each sensor the portion of its sensor area that covers it. The term *efficiency* denotes the efficiency of using a particular model and is denned as the ratio between the covered area and the portion used. The term *prob. to function* denotes the probability for the model to be functional. Assume that all the sensors, independent of their sensing range, have the probability p to fail, $0 \le p \le 1$, thus the probability to function for sensor is $1 - p$. Also we assume that any two failures are independent of one another.

The probabilities to function are obtained as follows.

In case of the minimum coverage model, this network functions only if all sensors are functional; the probability in this case is $(1 - p)^7$. Thus the probability to function is as follows:

$$P_{min.cov.} = (1-p)^7.$$

In case of the square model, this basic network functions if either all sensors are functional (the probability in this case is $(1 - p)^4$), or three sensors are functional and one is not (the probability in this case is $p(1 - p)^3$ and is multiplied by the number of such combinations which is 4) or two diagonally opposed sensors are functional and the other two are not (the probability in this case is $p^2(1 - p)^2$ and is multiplied by the number of such combinations which is 2). The probability to function is as below:

$$\begin{aligned} P_{square} &= (1-p)^4 + 4p(1-p)^3 + 2p^2(1-p)^2 \\ &= (1-p)^2(1+2p-p^2). \end{aligned}$$

In case of the hexagonal model, this basic network functions if either all sensors are functional (the probability in this case is $(1 - p)^6$), or five sensors are functional and one is not (the probability in this case is $p(1 - p)^5$ and is multiplied by the number of such combinations which is 6), or two non-consecutive sensors are not functional and the others are (the probability in this case is $p^2(1 - p)^4$ and is multiplied by the number of such combinations which is 15), or three non-consecutive sensors are not functional and the others are (the probability in this case is $p^3(1-p)^3$ and is multiplied by the number of such combinations which is 2). The probability to function is as follows:

$$P_{hexa} = (1-p)^6 + 6p(1-p)^5 + 15p^2(1-p)^4 + 2p^3(1-p)^3 = (1-p)^2(1+3p+6p^2-8p^3).$$

In case of the improved 8-node model, this basic network functions if either the larger range sensor is functional (the probability in this case is $1-p$) or the larger range sensor is not functional, but all the others are (the probability in this case is $p(1-p)^7$). The probability to function is as follows:

$$P_{imp} = (1-p) + p(1-p)^7.$$

From Table 1, we note that the minimal coverage model has the best efficiency, followed by the 8-node improved and hexagonal (the same value), and at last the square model.

Table 1 Comparisons among the four models

Model	*Area*	*Used*	*Efficiency*	*Prob. to function*
Square	r^2	Πr^2	$\frac{1}{\Pi} \simeq 0.318$	$(1-p)^2(1+2p-p^2)$
Hexagonal	$\frac{3r^2\sqrt{3}}{2}$	$2\Pi r^2$	$\frac{3\sqrt{3}}{4\Pi} \simeq 0.413$	$(1-p)^3(1+3p-2p^3)$
Min. cov.	$\frac{9\sqrt{3}r^2}{2}$	$3\Pi r^2$	$\frac{3\sqrt{3}}{2\Pi} \simeq 0.827$	$(1-p)^7$
Improved	$\frac{9\sqrt{3}r^2}{2}$	$6\Pi r^2$	$\frac{9\sqrt{3}}{12\Pi} \simeq 0.413$	$1-p+p(1-p)^7$

As far as the probability to function, the minimal model has the lowest value. For $0 \le p \le 0.082$, the square model is better than the hexagonal, which is better than the improved 8-node model. For $0.083 \le p \le 0.348$, the square model is better than the improved 8-node model, which is better than the hexagonal model. For $p \ge 0.349$, the improved 8-node model is better than the square model, which is better than the hexagonal model.

3.1 *Reliability function of the proposed models*

Reliability is one of the most important attributes of a system. Markov modelling is a widely used analytical technique for complex systems (Shooman, 1968; Trivedi, 1982; Johnson, 1989). It uses system state and state transitions. The state of the system comprises all it needs to be known to fully describe it at any given instant of time (Johnson, 1989). Each state of the Markov model is a unique combination of faulty and non-faulty modules. There is one state called ‘F’ which is the failed state (the system does not function anymore). A state transition occurs when one or more modules had failed or had recovered (if possible), and is characterised by probabilities (to fail or to recover).

The *exponential failure law* states that the reliability of a system varies exponentially as a function of time, for a constant failure rate function. The reliability of the system at time t, $R(t)$, is an exponential function of the failure rate, λ, which is a constant:

$$R(t) = e^{-\lambda t}$$

where λ is the constant failure rate.

If we assume that each module (in our case, each sensor) in the models we have proposed obeys the exponential failure law and has a constant failure rate of λ, the probability of each module (sensor) being failed at some time $t+\Delta t$, given that the module (sensor) was operational at time t, is given by $1-e^{-\lambda\Delta t}$. For small values of Δt, the expression reduces to $1-e^{-\lambda\Delta t} \approx \lambda\Delta t$. In other words, the probability that a sensor will fail within the time period Δt is approximately $\lambda\Delta t$.

We make following three assumptions that are commonly made in reliability:

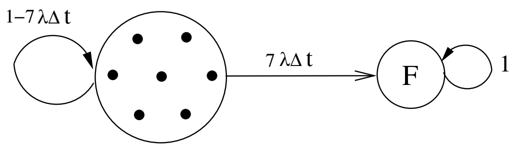
1 *A perfect state*: The system starts with no failures.

2 *A failure is permanent*: Once a module has failed, it does not recover.

3 *There is one failure at a time*: Two or more simultaneously failures are seen as sequential ones.

For a generic system state S, let $P_S(t)$ be the probability that the system is in state S at the moment time t. For each state S, the probability of the system being in that state at some time $t+\Delta t$ depends on the probability that the system was in that state S at time t and on any probability that the system was in another state S' and it has transitioned from S' to S.

Figure 5 Markov model of the minimal coverage model



3.2 *Minimum coverage model*

The minimum coverage model is 0-tolerant: The failure of a single node makes the network non-functional. The Markov model is shown in Figure 5. The single state in which all seven sensors are operational is named ‘7’. The failed state is named ‘F’ and is reached when any of the sensors fails. The probability of a sensor to fail within the time interval Δt is $\lambda\Delta t$. Thus the probability of the system to go from state ‘7’ to state ‘F’ is the probability of any sensor to fail multiplied by the number of such cases, which is ‘7’. The probability of the system to remain in state ‘7’ is 1 minus the probability of the system to go into any other state. Since state ‘F’ is the only other state reachable from state ‘7’, the probability of the system to remain in state ‘7’ is $1-7\lambda\Delta t$.

The equations of the Markov model of the minimum coverage model can be written from the state diagram shown in Figure 5 as follows.

The reliability of the system is the probability of the system to be in any of the non-failed states: $R_{min.cov.}(t) = 1 - P_F(t) = P_7(t)$. We obtain the system of equations below, with the initial values for $P_7(0)=1$ and $P_F(0)=0$. Taking the limit as Δt approaches 0 results in a set of differential equations to which Laplace transformation can be applied. Then we applied the reverse Laplace transformation.

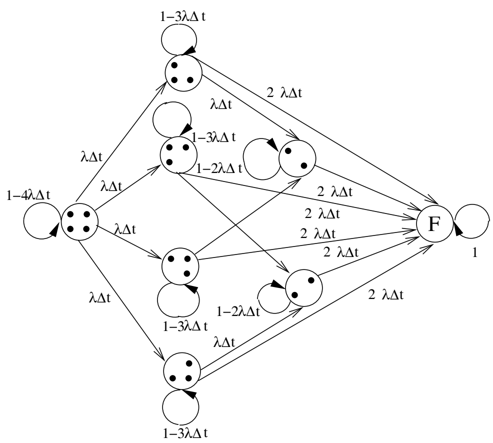
$$\begin{cases} P_7(t+\Delta t)=(1-7\lambda\Delta t)P_7(t) \\ P_F(t+\Delta t)=P_F(t)+7\lambda\Delta t)P_7(t) \end{cases} \Rightarrow \begin{cases} P_7(t)=e^{-7\lambda t} \\ P_F(t)=1-7e^{-7\lambda t} \end{cases}.$$

We obtain the reliability of the system to be $R_{min.cov.}(t)=e^{-7\lambda t}$.

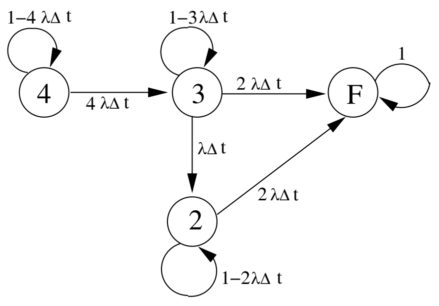
3.3 Square fault-tolerant model

The square fault-tolerant model is 1-fault tolerant: The system is still functional if a single node fails. If two adjacent nodes fail, the system becomes non-functional, but if two diagonally opposite nodes fail, the system remains functional. The Markov model of the square fault-tolerant model is shown in Figure 6(a). It can be reduced further as follows. The single state in which all four sensors are operational is named '4'. The four states in which a single sensor has failed can be reduced to one state named '3'. The two states in which two diagonally opposite sensors have failed can be reduced to one state named '2'. The reduced Markov model is shown in Figure 6(b).

Figure 6 Square model



(a) Markov Model



(b) Reduced Markov Model

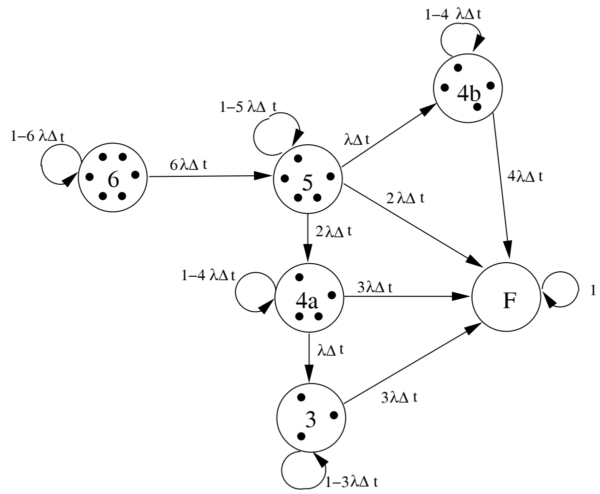
The reliability of the system is the probability of the system to be in any of the non-failed states: $R_{square}(t)=1-P_F(t)=P_4(t)+P_3(t)+P_2(t)$.

The equations of the Markov model of the square fault-tolerant model can be written from the reduced state diagram shown in Figure 6(b). Taking the limit as Δt approaches 0 results in a set of differential equations to which Laplace transformation can be applied. Then we applied the reverse Laplace transformation. We obtain the reliability of the system to be $R_{square}(t)=2e^{-2\lambda t}-e^{-4\lambda t}$ (see Section B.1 for more details).

3.4 Hexagonal fault-tolerant model

The Markov model of the hexagonal fault-tolerant model has more than 30 states. Some of the states are equivalent to one another, so we can reduce the number of states as follows. The single state in which all six sensors are operational is named '6'. The six states in which a sensor has failed can be reduced to one state named '5'. The states in which two sensors adjacent to a working sensor have failed can be reduced to one state named '4a'. The states in which two diagonally opposite sensors have failed can be reduced to one state named '4b'. The states in which three non-consecutive sensors have failed can be reduced to one state named '3'. The reduced Markov model is shown in Figure 7.

Figure 7 Reduced Markov model of the hexagonal model



The reliability of the system is the probability of the system to be in any of the non-failed states: $R_{hexagon}(t) = 1 - P_F(t) = P_6(t) + P_5(t) + P_{4a}(t) + P_{4b}(t) + P_3(t)$.

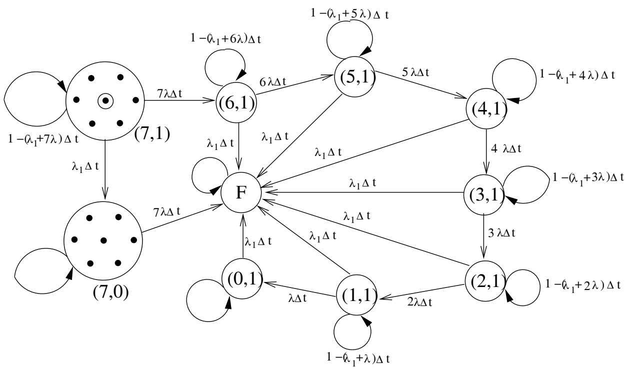
The equations of Markov model of the hexagonal fault-tolerant model can be written from the reduced state diagram shown in Figure 7. Taking the limit as Δt approaches 0 results in a set of differential equations to which Laplace transformation can be applied. We apply thereafter the reverse Laplace transformation. We obtain the reliability of the system to be $R_{hexagon}(t) = 2e^{-3\lambda t} + 3e^{-4\lambda t} - 6e^{-5\lambda t} + 2e^{-6\lambda t}$ (see Section B.2 for more details).

3.5 Improved 8-node model

The Markov model of the improved 8-node model has more than 40 states. Some of the states are equivalent to one another, so we can reduce the states as follows. We label the states by the number of sensors of each type, first the sensor whose sensing range is r, second the sensor whose sensing range is $r\sqrt{3}$. The single state in which all eight sensors are operational is named '(7,1)'. The state in which the $r\sqrt{3}$

sensor has failed is called '(7,0)'. The seven states in which a r sensor has failed can be reduced to one state named '(6,1)'. The states in which two r sensors have failed can be reduced to one state named '(5,1)'. The states in which three r sensors have failed can be reduced to one state named '(4,1)'. The states in which four r sensors have failed can be reduced to one state named '(3,1)'. The states in which five r sensors have failed can be reduced to one state named '(2,1)'. The states in which six r sensors have failed can be reduced to one state named '(1,1)'. The states in which all r sensors have failed can be reduced to one state named '(0,1)'. The reduced Markov model is shown in Figure 8.

Figure 8 Reduced Markov model of the improved 8-node model



We assume that the $r\sqrt{3}$ sensor has the constant failure rate of λ_1, while a r sensor has the constant failure rate of λ.

The reliability of the system is the probability of the system to be in any of the non-failed states:

$$R_{improved}(t) = 1 - P_F(t) = P_{(7,1)}(t) + P_{(7,0)}(t) + P_{(6,1)}(t) + P_{(5,1)}(t) + P_{(4,1)}(t) + P_{(3,1)}(t) + P_{(2,1)}(t) + P_{(1,1)}(t) + P_{(0,1)}(t).$$

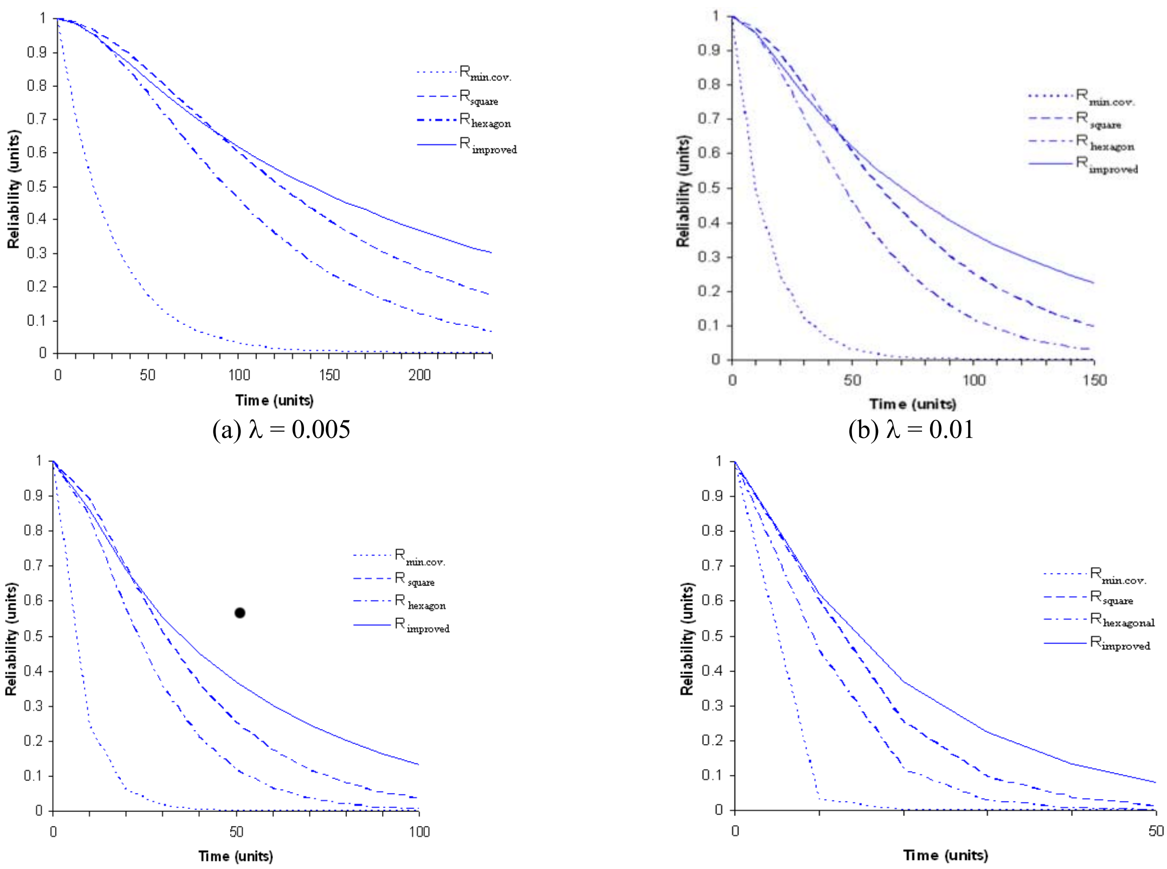
The equations of Markov model of the improved 8-node fault-tolerant model can be written from the reduced state diagram shown in Figure 8. Assuming $\lambda_1 = \lambda$ and taking the limit as Δt approaches 0 results in a set of differential equations to which the Laplace transformation can be applied. Thereafter we apply the reverse Laplace transformation. We obtain the reliability of the system to be $R_{improved}(t) = e^{-\lambda t} + e^{-7\lambda t} - e^{-8\lambda t}$ (see Section B.3 for more details).

3.6 Simulation results

We drew the reliability functions of the three models for values of $\lambda \in \{0.005, 0.01, 0.02, 0.05\}$ (Figure 9). As λ draws closer to 1, the difference between reliability values for the three models becomes smaller.

From Figure 9, we can observe that the minimum coverage model has the lowest reliability, followed by the hexagonal model. That makes the minimum coverage model the most unreliable, and the hexagonal model the most unreliable among the three 1-fault tolerant models.

Figure 9 Reliability chart (constant λ) (see online version for colours)



(a) $\lambda = 0.005$ (b) $\lambda = 0.01$

(c) $\lambda = 0.02$ (d) $\lambda = 0.05$

For any value of $\lambda \in \{0.005, 0.01, 0.02, 0.05\}$, at the beginning (*time* = 0), the square model has a better reliability value than the improved 8-node model. But over time, once the reliability values for the square and improved 8-node model drew closer to the value 0.7, the two models switch and remain so thereafter: The better reliability is achieved by the improved 8-node model, while the square model has a worse reliability.

We can conclude that in the long run, the improved 8-node model is more reliable among all models studied in this paper. This conclusion is consistent among all the four figures.

4 Conclusion

We have proposed three 1-fault tolerant models, and we have compared them among one another, and with the minimal coverage model (the uniform model expanded to seven nodes instead of three nodes). We note that the minimal coverage model has the best efficiency, followed by the 8-node improved and hexagonal (the same value), and the square model. As far as the probability to function, the minimal coverage model has the lowest value. For $0 \leq p \leq 0.082$, the square model is better than the hexagonal and the hexagonal is better than the improved 8-node model. For $0.083 \leq p \leq 0.348$, the square model is better than the improved 8-node model, and the improved 8-node model is better than the hexagonal model. For $p \geq 0.349$, the improved 8-node model is better than the square model, and the square model is better than the hexagonal model.

The most unreliable model among these models is the minimum coverage model; the most unreliable among the proposed 1-fault tolerant models is the hexagonal model. The improved 8-node model is the most reliable over time except at the very beginning, where the square model is more reliable. But after a short while, the improved 8-node model becomes and remains the better one.

We are currently working on algorithms to move sensors in order to preserve the network functionality when more than one fault occurs. If the network layout is composed of hundreds of patterns (as proposed here), in some cases sensors need to be moved to cover areas left uncovered by faulty or moving sensors.

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Notes

1 A shorter version of this paper was presented under the title ‘Fault tolerant coverage models for sensor networks’ at IOCS 2005.

Appendix

A Computing covered areas

In this section, we present major steps in computing areas covered by two, three or four sensors in each of the three proposed models.

A.1 Square model

Let A, B and C to be the disjoint areas as drawn in Figure 1(b). We have observed that $S_{2s}^{square} = 4S_A$, $S_{3s}^{square} = 8S_B$, $S_{4s}^{square} = 4S_C$, and we obtain that:

$$\begin{cases} S_A + 2S_B + S_C & = \dfrac{r^2}{4} \\ S_B + S_C + \dfrac{1}{4} & = \dfrac{\Pi r^2}{8} \\ 4S_B + 4S_C + S_A & = \dfrac{\Pi r^2}{3} - \dfrac{r^2\sqrt{3}}{4} \end{cases}$$

$$\Rightarrow \begin{cases} S_A & = r^2 - \dfrac{r^2\sqrt{3}}{4} - \dfrac{\Pi r^2}{6} \\ S_B & = -\dfrac{r^2}{2} + \dfrac{r^2\sqrt{3}}{4} + \dfrac{\Pi r^2}{24} \\ S_C & = \dfrac{r^2}{4} - \dfrac{r^2\sqrt{3}}{4} + \dfrac{\Pi r^2}{12} \end{cases}$$

$$\Rightarrow \begin{cases} S_{2s}^{square} & = 4r^2 - r^2\sqrt{3} - \dfrac{2\Pi r^2}{3} \\ S_{3s}^{square} & = -4r^2 + 2r^2\sqrt{3} + \dfrac{\Pi r^2}{3}. \\ S_{4s}^{square} & = r^2 - r^2\sqrt{3} + \dfrac{\Pi r^2}{3} \end{cases}$$

A.2 Hexagonal odel

Let A and B be the disjoint areas as shown in Figure 2(b). We observe that $S_{2s}^{hexagon} = 6S_A$ and $S_{3s}^{hexagon} = 6S_B$, and we obtain the following:

$$\begin{cases} S_A + S_B & = \dfrac{r^2\sqrt{3}}{4} \\ \dfrac{1}{2}S_B + \dfrac{r^2\sqrt{3}}{4} & = \dfrac{\Pi r^2}{6} \end{cases} \Rightarrow \begin{cases} S_A & = \dfrac{3r^2\sqrt{3}}{4} - \dfrac{\Pi r^2}{3} \\ S_B & = \dfrac{\Pi r^2}{3} - \dfrac{r^2\sqrt{3}}{2} \end{cases}$$

$$\Rightarrow \begin{cases} S_{2s}^{hexagon} & = \dfrac{9r^2\sqrt{3}}{2} - 2\Pi r^2 \\ S_{3s}^{hexagon} & = -3r^2\sqrt{3} + 2\Pi r^2 \end{cases}.$$

A.3 Improved 8-node model

Let A, B, C, D, E and F be some disjoint areas as shown in Figure 4. We observe that $S_{2s}^{improved} = S_F + 2S_D, S_{3s}^{improved} = 2S_E + S_C + 2S_B$ and $S_{4s}^{improved} = S_A$, and we obtain the following:

$$\begin{cases} S_A + 2S_B & = \dfrac{3\Pi r^2}{4} - \dfrac{3r^2\sqrt{3}}{4} \\ S_A + S_C & = \dfrac{\Pi r^2}{6} - \dfrac{r^2\sqrt{3}}{4} \\ S_A + 2S_B + S_C + 2S_D + 2S_E + S_F & = \dfrac{3r^2\sqrt{3}}{4} \\ S_A + S_B + S_C + S_D + S_E & = \dfrac{\Pi r^2}{6} \\ S_A + S_C & = S_E \\ 3S_E + 3S_F & = \dfrac{\left(r\sqrt{3}\right)^2\sqrt{3}}{4} \\ 2S_E + S_F & = \dfrac{\Pi r^2}{6} \end{cases}.$$

$$\Rightarrow \begin{cases} S_A & = \dfrac{\Pi r^2}{2} - \dfrac{3r^2\sqrt{3}}{4} \\ S_B & = \dfrac{\Pi r^2}{8} \\ S_C & = \dfrac{r^2\sqrt{3}}{2} - \dfrac{\Pi r^2}{3} \\ S_D & = \dfrac{r^2\sqrt{3}}{2} - \dfrac{7\Pi r^2}{24} \\ S_E & = \dfrac{\Pi r^2}{6} - \dfrac{r^2\sqrt{3}}{4} \\ S_F & = \dfrac{r^2\sqrt{3}}{2} - \dfrac{\Pi r^2}{6} \end{cases} \Rightarrow \begin{cases} S_{2s}^{improved} & = \dfrac{9r^2\sqrt{3}}{2} - 2\Pi r^2 \\ S_{3s}^{improved} & = 2\Pi r^2 - 3r^2\sqrt{3} \\ S_{4s}^{improved} & = \dfrac{\Pi r^2}{2} - \dfrac{3r^2\sqrt{3}}{4} \end{cases}$$

B Calculating the reliability

In this section, we present the major steps in computing the reliability function of the three proposed models. We assume that each sensor obeys the exponential failure law and has a constant failure rate of λ. The probability that a sensor fails within time period Δt is approximately $\lambda\Delta t$.

For each state S, the probability of the system being in that state at some time $t + \Delta t$ depends on the probability that the system was in state S at time t and on any probability that the system was in another state S' and it has a transition from S' to S.

B.1 Square model

The reliability of the system is the probability of the system to be in any of the non-failed states: $R_{square}(t) = 1 - P_F(t) = P_4(t) + P_3(t) + P_2(t)$.

The initial values for $P_4(0) = 1, P_3(0) = P_2(0) = P_F(0) = 0$.

The equations of the Markov model of the square fault-tolerant model can be written from the reduced state diagram shown in Figure 6(b) as follows. Taking the limit as Δt approaches 0 results in a set of differential equations to which Laplace transformation can be applied. Then we applied the reverse Laplace transformation.

$$\begin{cases} P_4(t+\Delta t) &= (1-4\lambda\Delta t)P_4(t) \\ P_3(t+\Delta t) &= (1-3\lambda\Delta t)P_3(t) + 4\lambda\Delta t P_4(t) \\ P_2(t+\Delta t) &= (1-2\lambda\Delta t)P_2(t) + \lambda\Delta t P_3(t) \\ P_F(t+\Delta t) &= P_F(t) + 2\lambda\Delta t)P_3(t) + 2\lambda\Delta t P_2(t) \end{cases}$$

$$\Rightarrow \begin{cases} P_4(t) &= e^{-4\lambda t} \\ P_3(t) &= 4e^{-3\lambda t} - 4e^{-4\lambda t} \\ P_2(t) &= 2e^{-2\lambda t} - 4e^{-3\lambda t} + 2e^{-4\lambda t} \\ P_F(t) &= 1 - 2e^{-2\lambda t} + e^{-4\lambda t} \end{cases}.$$

The reliability is $R_{square}(t) = 2e^{-2\lambda t} - e^{-4\lambda t}$.

B.2 Hexagonal model

The reliability of the system is the probability of the system to be in any of the non-failed states: $R_{hexagon}(t) = 1 - P_F(t)$ $= P_6(t) + P_5(t) + P_{4a}(t) + P_{4b}(t) + P_3(t)$.

The equations of Markov model of the hexagonal fault-tolerant model can be written from the reduced state diagram shown in Figure 7 as follows.

The initial values for $P_6(0) = 1$, $P_5(0) = P_{4a}(0) = P_{4b}(0) = P_3(0) = P_F(0) = 0$. Taking the limit as Δt approaches 0 results in a set of differential equations to which Laplace transformation can be applied. We apply thereafter the reverse Laplace transformation.

$$\begin{cases} P_6(t+\Delta t) &= (1-6\lambda\Delta t)P_6(t) \\ P_5(t+\Delta t) &= (1-5\lambda\Delta t)P_5(t) + 6\lambda\Delta t P_6(t) \\ P_{4a}(t+\Delta t) &= (1-4\lambda\Delta t)P_{4a}(t) + 2\lambda\Delta t P_5(t) \\ P_{4b}(t+\Delta t) &= (1-4\lambda\Delta t)P_{4b}(t) + \lambda\Delta t P_5(t) \\ P_3(t+\Delta t) &= (1-3\lambda\Delta t)P_3(t) + \lambda\Delta t P_{4a}(t) \\ P_F(t+\Delta t) &= P_F(t) + \lambda\Delta t(2P_5(t) + 3P_{4a}(t) + 4P_{4b}(t) + 3P_3(t)) \end{cases}$$

$$\Rightarrow \begin{cases} P_6(t) &= e^{-6\lambda t} \\ P_5(t) &= 6e^{-5\lambda t} - 6e^{-6\lambda t} \\ P_{4a}(t) &= 6e^{-4\lambda t} - 12e^{-5\lambda t} + 6e^{-6\lambda t} \\ P_{4b}(t) &= 3e^{-4\lambda t} - 6e^{-5\lambda t} + 3e^{-6\lambda t} \\ P_3(t) &= 2e^{-3\lambda t} - 6e^{-4\lambda t} + 6e^{-5\lambda t} - 2e^{-6\lambda t} \\ P_F(t) &= 1 - 2e^{-3\lambda t} + e^{-4\lambda t} + 6e^{-5\lambda t} - 2e^{-6\lambda t} \end{cases}.$$

The reliability is $R_{hexagon}(t) = 2e^{-3\lambda t} + 3e^{-4\lambda t} - 6e^{-5\lambda t} + 2e^{-6\lambda t}$.

B.3 Improved 8-node model

The reliability of the system is the probability of the system to be in any of the non-failed states: $R_{improved}(t) = 1 - P_F(t) = P_{(7,1)}(t) + P_{(7,0)}(t) + P_{(6,1)}(t) + P_{(5,1)}(t) + P_{(4,1)}(t) + P_{(3,1)}(t) + P_{(2,1)}(t) + P_{(1,1)}(t) + P_{(0,1)}(t)$.

We assume that the $r\sqrt{3}$ sensor has the constant failure rate of λ_1, while a r sensor has the constant failure rate of λ.

The equations of Markov model of the improved 8-node fault-tolerant model can be written from the reduced state diagram shown in Figure 8 as follows.

The initial values for $P_{(7,1)}(0) = 1$, $P_{(7,0)}(0) = P_{(6,1)}(0) = P_{(5,1)}(0) = P_{(4,1)}(0) = P_{(3,1)}(0) = P_{(2,1)}(0) = P_{(1,1)}(0) = P_{(0,1)}(0) = P_F(0) = 0$.

$$\begin{cases} P_{(7,1)}(t+\Delta t) &= (1-(\lambda_1+7\lambda)\Delta t)P_{(7,1)}(t) \\ P_{(7,0)}(t+\Delta t) &= (1-7\lambda\Delta t)P_{(7,0)}(t) + \lambda_1\Delta t)P_{(7,1)}(t) \\ P_{(6,1)}(t+\Delta t) &= (1-(\lambda_1+6\lambda)\Delta t)P_{(6,1)}(t) + 7\lambda\Delta t P_{(7,1)}(t) \\ P_{(5,1)}(t+\Delta t) &= (1-(\lambda_1+5\lambda)\Delta t)P_{(5,1)}(t) + 6\lambda\Delta t P_{(6,1)}(t) \\ P_{(4,1)}(t+\Delta t) &= (1-(\lambda_1+4\lambda)\Delta t)P_{(4,1)}(t) + 5\lambda\Delta t P_{5,1)}(t) \\ P_{(3,1)}(t+\Delta t) &= (1-(\lambda_1+3\lambda)\Delta t)P_{(3,1)}(t) + 4\lambda\Delta t P_{(4,1)}(t) \\ P_{(2,1)}(t+\Delta t) &= (1-(\lambda_1+2\lambda)\Delta t)P_{(2,1)}(t) + 3\lambda\Delta t P_{(3,1)}(t) \\ P_{(1,1)}(t+\Delta t) &= (1-(\lambda_1+\lambda)\Delta t)P_{(1,1)}(t) + 2\lambda\Delta t P_{(2,1)}(t) \\ P_{(0,1)}(t+\Delta t) &= (1-\lambda_1\Delta t)P_{(0,1)}(t) + \lambda\Delta t P_{(1,1)}(t) \\ P_F(t+\Delta t) &= P_F(t) + 7\lambda\Delta t P_{(7,0)}(t) + \lambda_1\Delta t(P_{(6,1)}(t) \\ &\quad + P_{(5,1)}(t) + P_{(4,1)}(t) + P_{(3,1)}(t) + P_{(2,1)}(t) \\ &\quad + P_{(1,1)}(t) + P_{(0,1)}(t)) \end{cases}.$$

Assuming $\lambda_1 = \lambda$, and taking the limit as Δt approaches 0 results in a set of differential equations to which the Laplace transformation can be applied. Thereafter we apply the reverse Laplace transformation.

$$\begin{cases} P_{(7,1)}(t) &= e^{-8\lambda t} \\ P_{(7,0)}(t) &= e^{-7\lambda t} - e^{-8\lambda t} \\ P_{(6,1)}(t) &= 7e^{-7\lambda t} - 7e^{-8\lambda t} \\ P_{(5,1)}(t) &= 21e^{-6\lambda t} - 42e^{-7\lambda t} + 21e^{-8\lambda t} \\ P_{(4,1)}(t) &= 35e^{-5\lambda t} - 105e^{-6\lambda t} + 105e^{-7\lambda t} - 35e^{-8\lambda t} \\ P_{(3,1)}(t) &= 35e^{-4\lambda t} - 140e^{-5\lambda t} + 210e^{-6\lambda t} \\ &\quad - 140e^{-7\lambda t} + 35e^{-8\lambda t} \\ P_{(2,1)}(t) &= 21e^{-3\lambda t} - 105e^{-4\lambda t} + 210e^{-5\lambda t} \\ &\quad - 210e^{-6\lambda t} + 105e^{-7\lambda t} - 21e^{-8\lambda t} \\ P_{(1,1)}(t) &= 7e^{-2\lambda t} - 42e^{-3\lambda t} + 105e^{-4\lambda t} \\ &\quad - 140e^{-5\lambda t} + 105e^{-6\lambda t} - 42e^{-7\lambda t} + 7e^{-8\lambda t} \\ P_{(0,1)}(t) &= e^{-\lambda t} - 7e^{-2\lambda t} + 21e^{-3\lambda t} - 35e^{-4\lambda t} + 35e^{-5\lambda t} \\ &\quad - 21e^{-6\lambda t} + 7e^{-7\lambda t} - e^{-8\lambda t} \\ P_F(t) &= 1 - e^{-\lambda t} - e^{-7\lambda t} + e^{-8\lambda t} \end{cases}.$$

The reliability is $R_{improved}(t) = e^{-\lambda t} + e^{-7\lambda t} - e^{-8\lambda t}$.