

Ice Cream Problem

(1996 Dagstuhl)

An ice-cream maker has two states:

V and C

Two tasks:

V = "Need vanilla"

C = "Need chocolate"

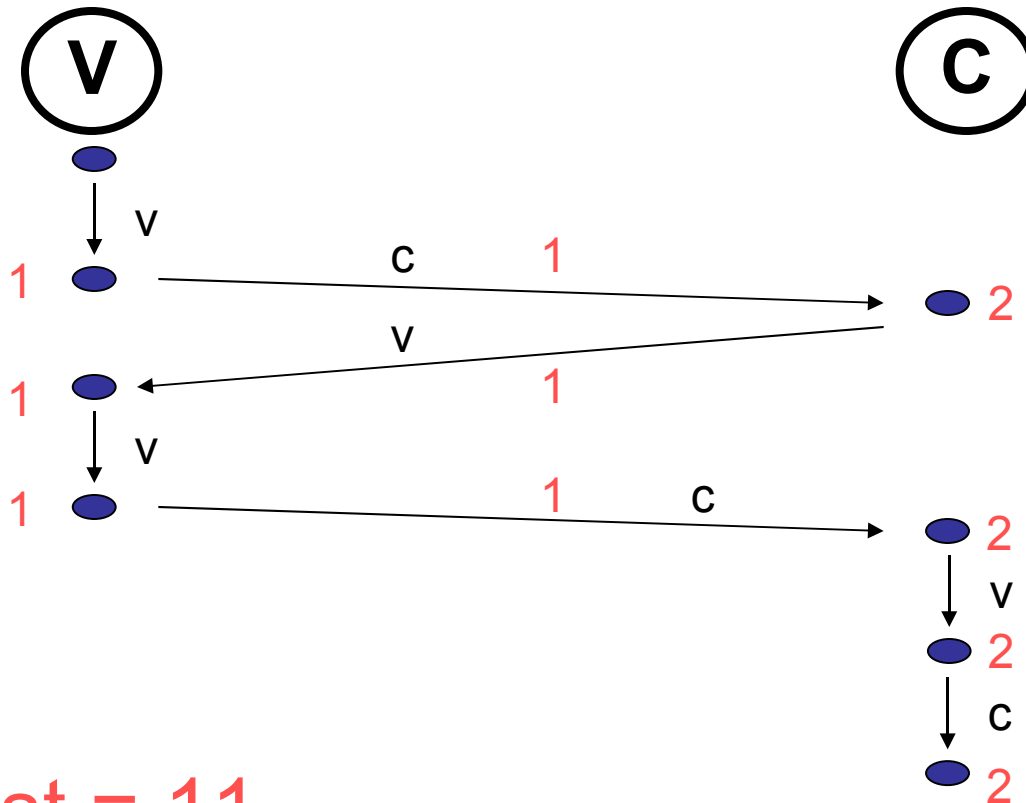
It costs 1 to change state

	V	C
V	1	2
C	4	2

Given a sequence of tasks: How do we serve them?

Assume start state = v

Task sequence = $vcvvcvc$



Opt Cost = 11

Metric Space $M = \{x_1, \dots, x_k\}$

d_{ij} = distance (x_i, x_j)

A task is a vector $r = (r_1, \dots, r_k)$

1. The current state is some x_i
2. Request r is received
3. Change state to some x_j
4. “Serve” the task
5. Change state to some x_ℓ

$$\text{Cost} = d_{ij} + r_i + d_{j\ell}$$

Competitiveness of a MTS

$$\overline{\lim} \frac{\text{online cost}}{\text{offline cost}} = C$$

The ice-cream problem:

$$C_{\text{det}} = \frac{7}{6} \text{ (Dag 1996)}$$

$$C_{\text{rand}} = \frac{12}{11}$$

Theorem: For any M of cardinality k , the unrestricted MTS on M is $2k-1$ competitive.

(deterministic)

Borodin, Linial, Saks (1987)

Randomized Competitiveness:
little is known.

Unrestricted Metrical Task System

Given metric space M

All non-negative vectors are tasks.

Without loss of generality:

All tasks are of the form

$$(0, 0, \dots, \underset{\substack{\uparrow \\ j^{\text{th}}}}{\varepsilon}, \dots, 0)$$

Where $\varepsilon > 0$ is arbitrarily small.

UMTS_k =

unrestricted MTS on
a uniform space of k points

(All distances = 1)



What is C_{rand} (against the oblivious adversary) ?

For $k = 2$, $C_{\text{rand}} = 2$.

Our Results

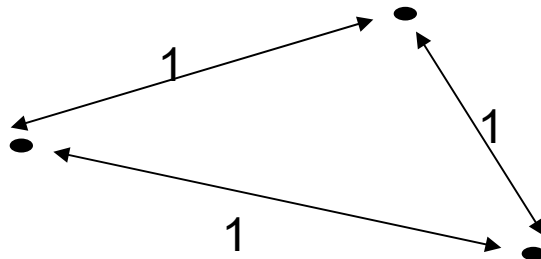
$$C \leq \frac{3}{2} H_k - \frac{1}{2^k} \text{ all } k$$

(generalizes Chrobak - Noga)

for $k = 2$, 2

for $k = 3$, $\frac{31}{12} = 2.58333\dots$

for $k = 4$, 3



Work Function for UMTS_k

$$w = (w_1, \dots, w_k) \quad |w_i - w_j| \leq 1$$

w_i = opt cost of serving sequence & ending in state i

Updating Work Function (Dynamic P.)

$w = (w_1, \dots, w_k)$ current work function

$r = (r_1, \dots, r_k)$ *next task*

$w \wedge r = \hat{w} = (\hat{w}_1, \dots, \hat{w}_k)$ *next work function*

For all i do: $\hat{w}_i := w_i + r_i;$

For all i, j do:

if $\hat{w}_i > \hat{w}_j + 1$ then
 $\hat{w}_i := \hat{w}_j + 1$

We say w is an offset function if $\min w_i = 0$.

Randomized Online Algorithm for UMTS₂

$$w = (w_1, w_2) \xrightarrow{a_2} (P_1, P_2)$$

$$\parallel$$

$$P_1 [x_1] + P_2 [x_2]$$

Defined by:

$$P_1 = \frac{1 - w_1 + w_2}{2}$$

$$P_2 = \frac{1 - w_1 + w_2}{2}$$



a_2 is 2 complete [Chrobak, Noga]

A Stable Distribution Algorithm for UMTS_k

Def: distribution = probability distribution on states.

\mathcal{W} = all work functions

\mathcal{D} = all distribution

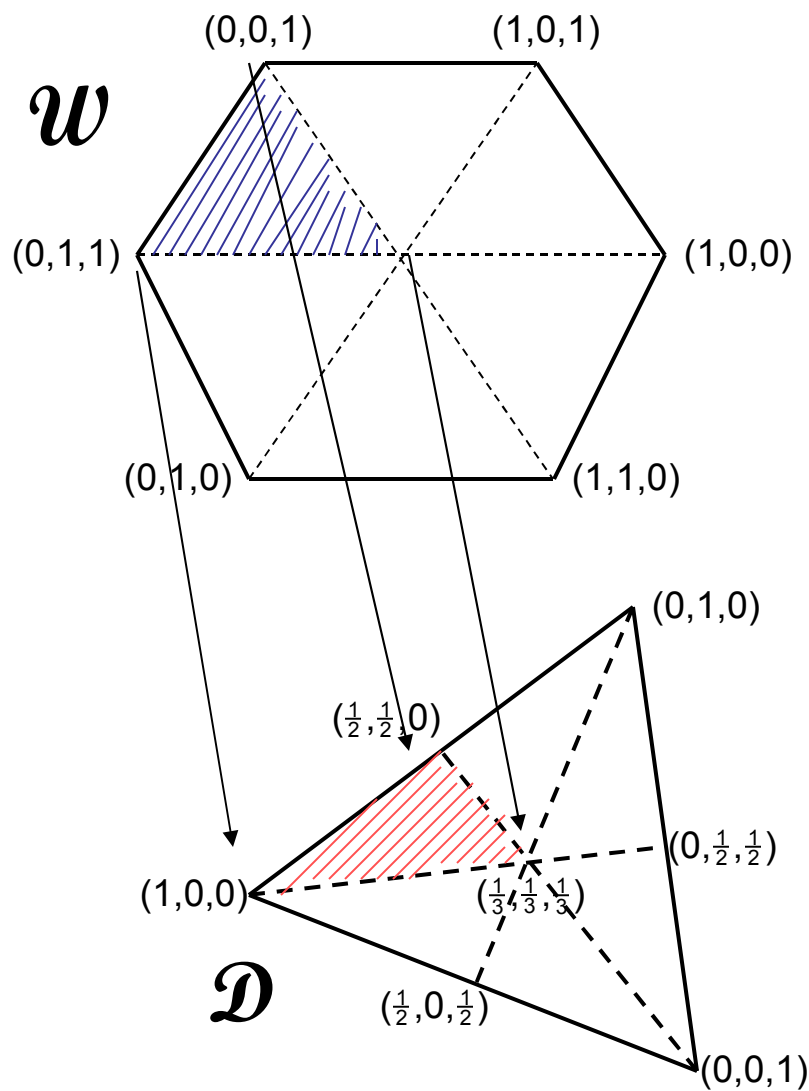
$$\mathcal{W} \xrightarrow{a} \mathcal{D}$$

Compute $w^t = w^{t-1} \wedge r^t$

“move” to a (w^t)

\mathcal{a}_k for UMTS $_k$, $k \geq 3$

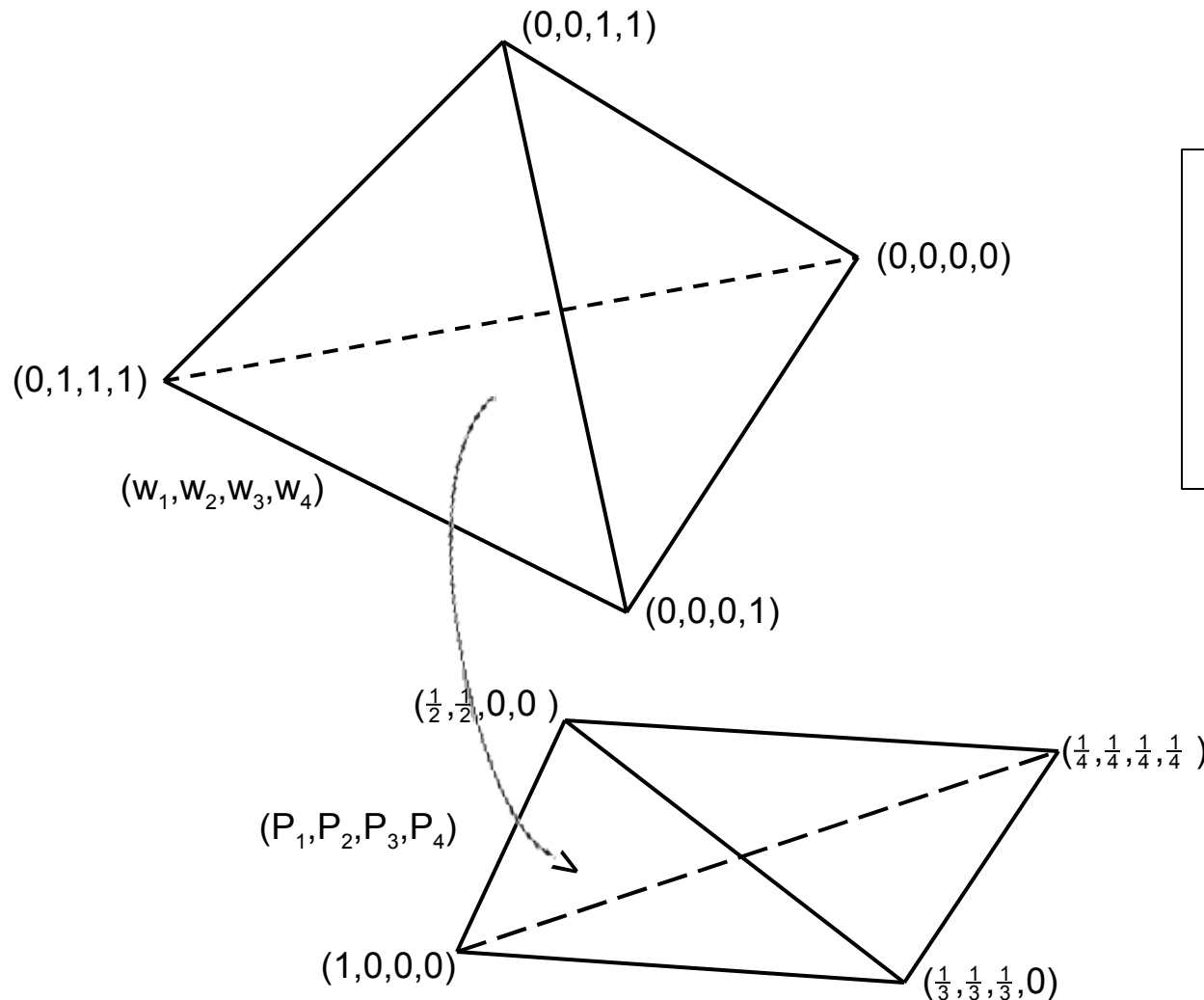
[Chrobak, Noga]



PL –
k! pieces

Define \mathbf{a}_k on one piece:

$$\Delta = \{w \in \mathcal{W} \mid w_1 \leq w_2 \leq \dots \leq w_k \leq w_1 + 1\}$$



$$l_i = w_{i+1} - w_1 \quad \text{if } i < k$$

$$l_k = w_1 + 1 - w_k$$

$$P_i = \sum_{j=1}^k \frac{1}{j} l_j$$

N.B. $P_i \geq 0$

$$\sum P_i = 1$$

\mathbf{a}_k is C competitive, $C = \frac{3}{2} H_k - \frac{1}{2} c_k$

$$W = (w_1, \dots, w_i, \dots, w_k) \longrightarrow P = (P_1, \dots, P_2)$$

$(0, \dots, \varepsilon, \dots, 0)$

$\|P, \hat{P}\|$

$$\hat{W} = (w_1, \dots, w_i + \varepsilon, \dots, w_k) \longrightarrow \hat{P} = (\hat{P}_1, \dots, \hat{P}_k)$$

Our cost =

$$\|P, \hat{P}\| + \varepsilon \cdot \hat{P}_i$$

We will define a potential Φ

$$1) \Phi(w + \lambda) = \Phi(w) - \lambda \cdot C$$

$$2) \Phi(w) - \Phi(\hat{w}) \geq \|P, \hat{P}\| + \varepsilon \cdot \hat{P}_i$$

On $\Delta = \{w \mid w_1 \leq w_2 \leq \dots \leq w_k \leq w_1 + 1\}$

Let $\Phi =$

$$\frac{1}{2} \sum_{1 \leq i < j \leq k} \frac{1}{j} l_i l_j + \frac{1}{4} \sum_{i=1}^k l_i^2 - \frac{3}{2} \sum_{i=1}^k \frac{1}{i} w_i + \frac{1}{2k} w_1$$

$$l_i = l_i(w) = \begin{cases} w_{i+1} - w_i & \text{if } i < k \\ w_1 + 1 - w_k & \text{if } i = k \end{cases}$$

Need to verify:

1) $\frac{\partial \Phi}{\partial w_i} \leq 0, \frac{\partial \Phi}{\partial w_i} \geq 0$ if $j \neq i$

2) Φ "fits" on boundary of Δ

3) $\Phi(w + \lambda) = \Phi(w) - \lambda \cdot C$

4) $\frac{\partial \Phi}{\partial w_i} + P_i - \frac{\partial P_i}{\partial w_i} \leq 0$

Lower Bounds for $UMTS_k$

Suppose $UMTS_k$ is C – competitive. Then \exists potential

$$\Phi_p^w \quad \begin{array}{l} \text{any workfunction } W \\ \text{any distribution } P \end{array}$$

s.t.

$$1) \quad \Phi_p^{w+\lambda} = \Phi_p^w - \lambda C$$

$$2) \quad \forall P \forall r \exists \hat{P} \quad \Phi_p^w \geq \Phi_{\hat{p}}^{w \wedge r} + \|P, \hat{P}\| + r \cdot \hat{P}$$

$$3) \quad |\Phi_p^w - \Phi_q^w| \leq \|P, Q\| \quad (1 - Lipschitz)$$

Variables:

$C, \Phi_{[i]}^w$ for all i , all special w .

Minimize C s.t.

$$\Phi_{[i]}^w = \Phi_{[i]}^{w+1/n} + 1/n C$$

If $w \wedge \zeta_i = \hat{w}_i$

$$\Phi_{[i]}^w \geq (1 - 1/n) \Phi_{[i]}^{\hat{w}} + 1/kn \sum_{j=1}^k \Phi_{[j]}^{\hat{w}} + 1/n$$

The resulting C will be a lower bound on the competitiveness of $UMTS_k$

LP lower bound proof

Fix $n \geq 1$

Def: $w = (w_1, \dots, w_k)$ is special

if 1) $\forall i \quad w_i = \frac{\text{integer}}{n}$

2) $\min_i w_i \in \{0, 1/n\}$

Def: $\zeta_i = (0, \dots, \underset{\substack{\uparrow \\ \mathfrak{t}^{\text{th}}}}{1/n}, \dots, 0)$ special tasks

N.B. If w is special and $\min w = 0$

and $\hat{w} = w \wedge \zeta_i$ then \hat{w} is special

Result $k = 3$

$n = 1$	$C \geq 1.833333333$
$n = 2$	$C \geq 2.16579413$
$n = 3$	$C \geq 2.28480506$
$n = 4$	$C \geq 2.34557796$
\vdots	
$n = 145$	$C \geq 2.52721024$

Conjecture: Converges to the true competitiveness

? $C \approx 2.53$?
? ?

Future work

1. LP for upper bounds
(seems straight forward, but there are issues.)
2. Convergence Conjecture as $n \longrightarrow \infty$
3. Non-uniform spaces,
non-universal MTS's
4. Differential Equation to obtain exact solutions.