The 3-Server Problem in the Plane

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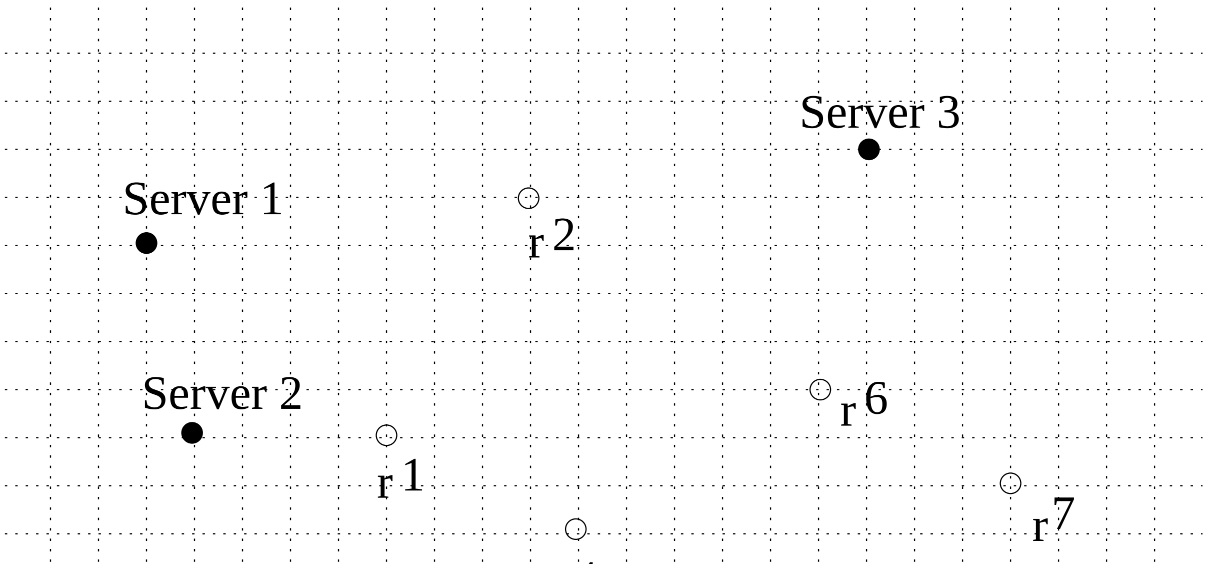
The k-Server Problem

k identical mobile servers in a metric space M

requests $r^1, r^2, \ldots$ are called out (by an adversary)

Our algorithm $\mathcal{A}$ must serve the requests with any one server

$\mathcal{A}$ is online



Competitiveness

For request sequence $\varrho = r^1, r^2, \ldots$ consider

cost (ϱ): the cost on ϱ achieved by the servers of our online algorithm $\mathcal{A}$

cost$_{opt}(\varrho)$: the cost on ϱ achieved by the server of an optimal offline algorithm

We say that our algorithm is C-*competitive* if for each sequence ϱ we have

cost $(\varrho) \leq C \cdot$ *cost*$_{opt}(\varrho) + B$

Important previous results

Lower bound of k

The k-Server Conjecture:
k competitive [Manasse, McGeoch, Sleator, 88]

2-competitive algorithm for $k = 2$

no trackless 2-competitive algorithm for $k = 2$.
[Bein, Larmore, 99]

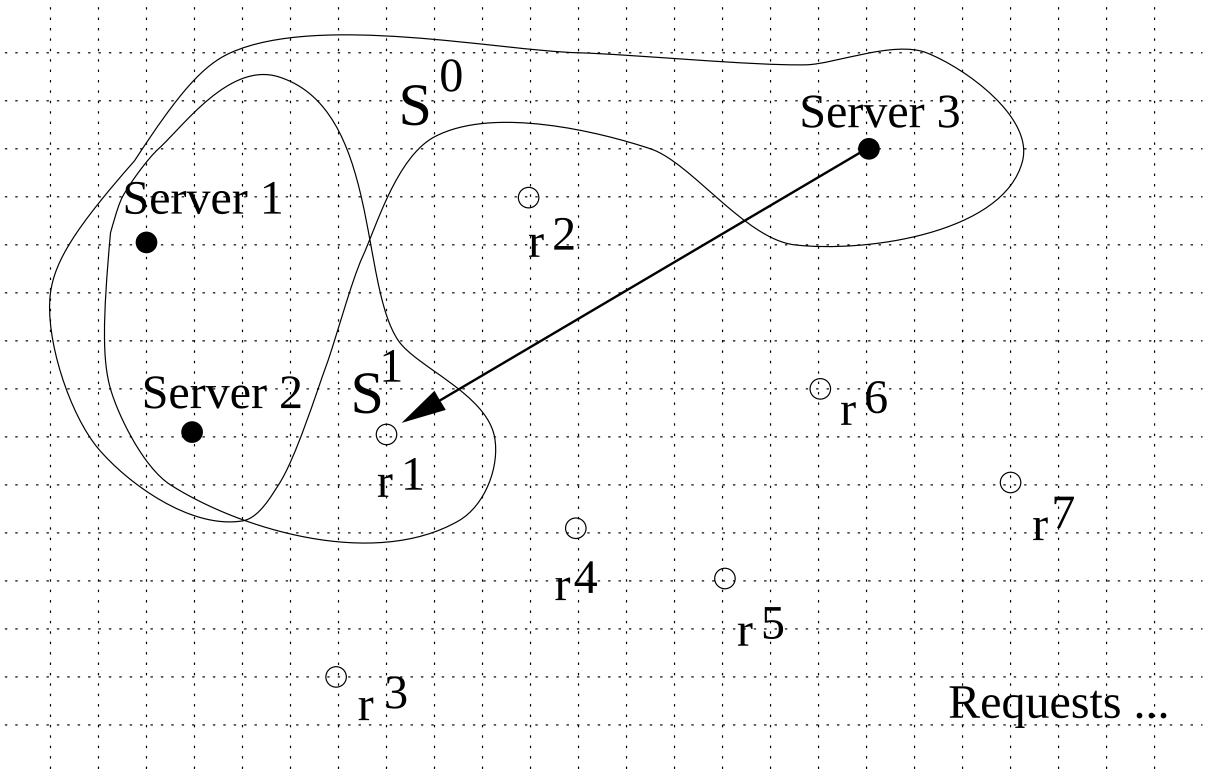
k-competitive for $k + 1$ points

k-competitive for trees [Chrobak, Larmore, 91]

Upper bound $2k-1$ [Koutsoupias, Papadimitriou 1994]

Upper bound k for $k + 2$ points

Configurations



Request are serviced by transitions from configuration to configuration $S^0, S^1, S^2, \ldots, S^t$

Work Functions

Given request sequence $\varrho = r^1 \ldots r^t \ldots$

$$\omega^t(S) \quad = \quad \text{optimal cost of servicing } \varrho$$
$$\text{while ending up in configuration } S$$

$\omega^t(S)$ can be calculated by dynamic programming:

$$\omega^t(S) = \min_{X \ni r} \left\{ \omega^{t-1}(X) + XS \right\}$$

Two algorithms that fail to be competitive

- ”Move Closest Server”
  = move the server which is closest to the request point
  = move the server at point x such that $|(S - x + r)S|$ is minimized

- Retrospect Greedy
  = move the server that “mimics” the optimal offline algorithm
  = move the server at point x such that $\omega(S - x + r)$ is minimized.

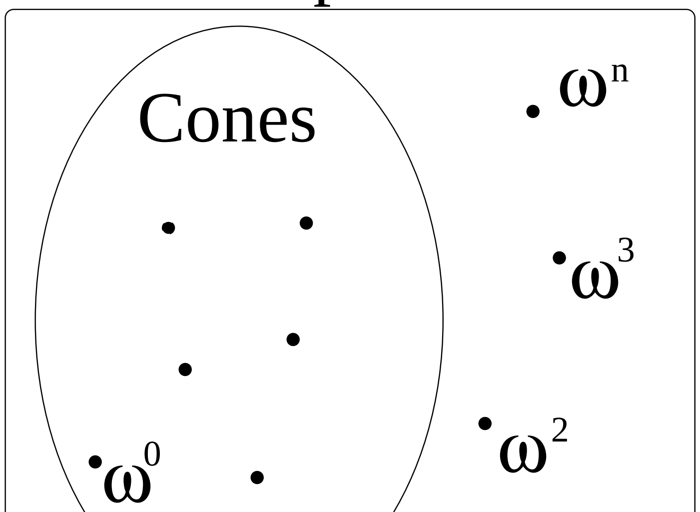
Function Space

$\{\{k -$ tuples $\to R \;\; = \mathcal{F}$

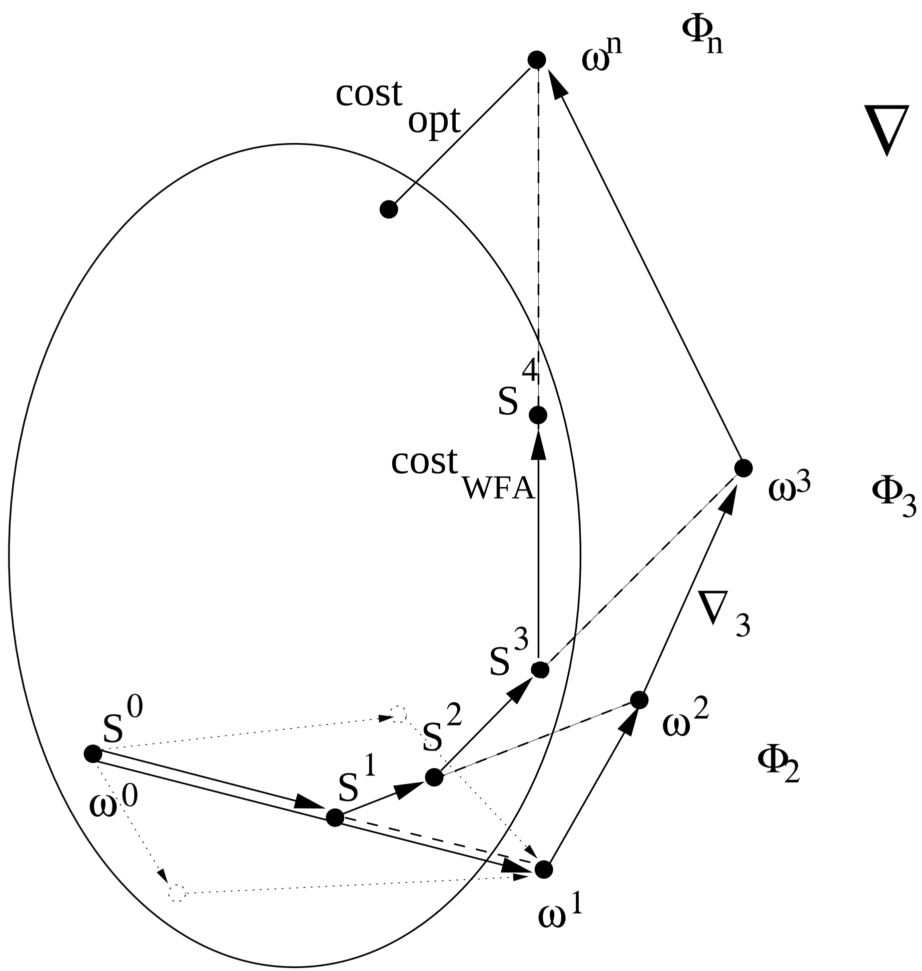
Use sup-norm metric $\|f, \;\; \| = \max_S\{|f(S) - \;\; (S)|$

Cone: $\chi_S(X) = \|S, X\|$

Function Space



WFA: "Move to best configuration"



$$cost_{\mathrm{WFA}} < \nabla - cost_{opt}$$

Potential

need to find Φ_t

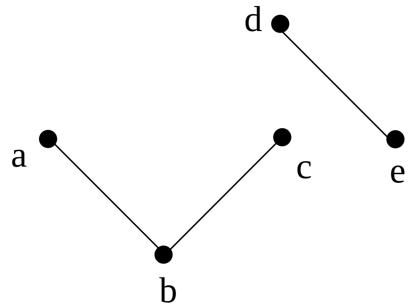
such that

- (UPDATE)
  $\Phi_t + \nabla_t \leq \Phi_{t-1}$

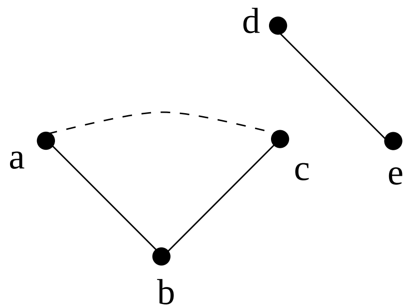
- (OFFSET)
  $\Phi_n \geq -(C+1)\min(\omega^n)$
  (easy!)

Therefore, by high school algebra...

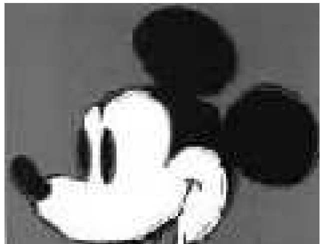
Mickey's Formula Primer



Mickey says: "It's 1"

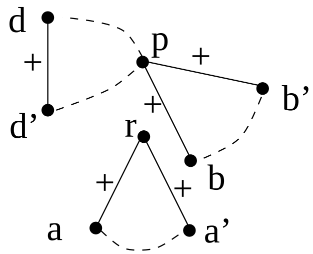


Mickey says: " It's $1 - \omega(rac)$"

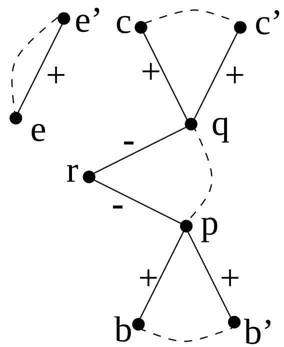


Searching for a Potential

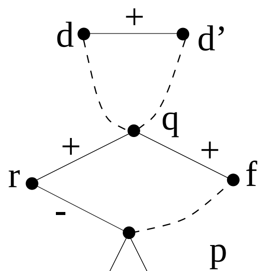
Ψ:



Λ:



Γ:



How to come up with these formulas...?

Ψ – the lazy potential

- assume current work function be ω, last request r
- adversary in some configuration X with $r \in X$
- following r assume a “worst-case” sequence $\xi = s^1 \ldots s^m$
- assume adversary will only make requests $s^i \in X$
- resulting work function χ will be *cone* on X
- therefore we conclude that $\Phi_\omega \geq \nabla_\xi(\omega) + \Phi_\chi$
- for any cone χ of configuration X, $\Phi_\chi = \Sigma X - 4 \cdot \chi(X)$
- thus our estimate is $\nabla_\xi(\omega) + \Sigma X - 4 \cdot \omega(X)$

$\widehat{\Psi}$ – the semi-lazy potential

<u>Larmore, Chrobak, DIMACS, 1991</u>

They show:

1. For the k server problem, if $k = 2$, then $\Psi = \widehat{\Psi}$ holds in every space.

2. If $\Psi = \widehat{\Psi}$ for k, $k \geq 3$ then the k server conjecture is true.

This talk:

- Consider $k = 3$, then there is a counter example with $\Psi < \widehat{\Psi}$ in a space of only 7 points

Main Result: Ψ is a potential in the Manhattan plane

OFFSET is easy...

UPDATE ....

We show

1. $\Psi_t + \nabla_t \leq \widehat{\Psi}^{t-1}$

2. $\widehat{\Psi} = \Psi$ in the Manhattan plane

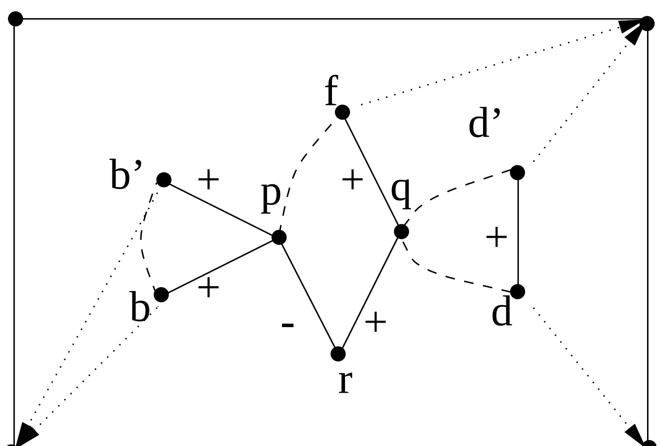
How does Manhattan help

We can show

- $\widehat{\Psi} \leq \Psi$ for 6 point spaces

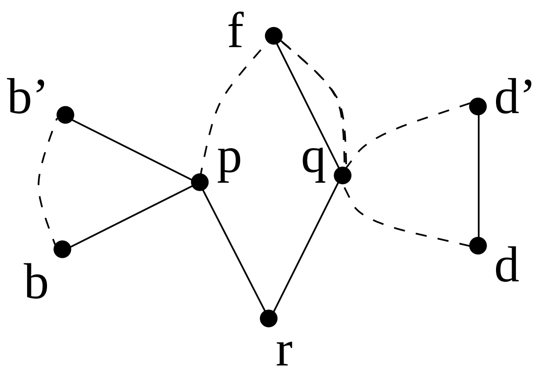
- $\widehat{\Psi} \leq \Psi$ for Manhattan plane

Common idea: identify points



Proof of one of the 50, or so, cases

Γ



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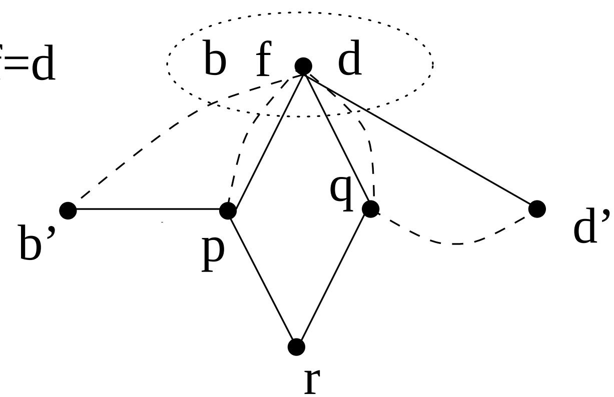
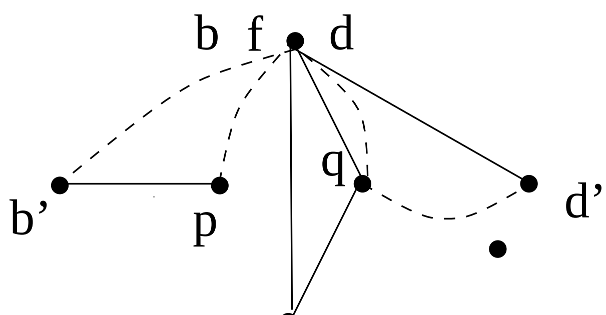
Γ



Ψ



A result for the Euclidean plane

Corollary 1 *WFA is* $3\sqrt{2}$*-competitive for 3 servers in the Euclidean plane*

Proof: Euclidean distance approximates the Manhattan distance with distortion $\sqrt{2}$.

Further directions

Conjecture: Lazy Φ works on circle, Euclidean plane, other ...

but ...

does not work in general