Chapter 1: Differential Form of Basic Equations
Lecture 03 Outline

• Some Comments on Shear Strain
• Material Laws
• Equilibrium
• Boundary Conditions
• Governing Equations
What Is the Difference Between Engineering Strain and Tensorial Strain

• Engineering and Tensorial Normal strains are defined the same:
  \( \varepsilon_{xx} = \varepsilon_{11} = \Delta L/L \)

• Engineering Shear strain, \( \gamma_{xy} \), is defined as the change in angle of two perpendicular axes as a material is deformed in shear.

• Tensorial shear strain is defined as \( \frac{1}{2} \) of engineering shear strain:
  \[
  \varepsilon_{12} = \frac{\gamma_{12}}{2} \quad \text{and} \quad \varepsilon_{xy} = \frac{\gamma_{xy}}{2}
  \]
Engineering Strain and Tensorial Strain

• It is sometimes convenient to write related equations in matrix form so that they are more compact.

• Albert Einstein devised a shorthand notation, *Einstein’s Indicial Notation*, to be able to write equations in an even more efficient manner (but we will not be using that here).

• All terms written with matrix notation should be exactly the same as if all the equations were written out explicitly.

• **Example:**

Strain Energy Density, \( W \), is defined as the area under the stress-strain curve. In the linear region of the curve, the area forms a triangle so:

\[
W = \frac{1}{2} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy})
\]

This example assumes a 2-D problem with applied normal stresses in the x- and y- directions, and an applied shear stress.
Engineering Strain and Tensorial Strain

• Strain Energy Density

\[ W = \frac{1}{2} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}) \]

This is the correct definition of \( W \).

• We would like to be able to define strain energy density in terms of matrix operations: \( W = \frac{1}{2} [\sigma][\varepsilon] \)

  – Using engineering strain

\[
W = \frac{1}{2} \begin{bmatrix}
\sigma_x & \tau_{xy} \\
\tau_{xy} & \sigma_y
\end{bmatrix}\begin{bmatrix}
\varepsilon_x & \gamma_{xy} \\
\gamma_{xy} & \varepsilon_y
\end{bmatrix} = \frac{1}{2} (\sigma_x \varepsilon_x + 2 \tau_{xy} \gamma_{xy} + \sigma_y \varepsilon_y)
\]

Using engineering strain in matrix equations does not give the correct \( W \).

  – Using tensorial strain

\[
W = \frac{1}{2} \begin{bmatrix}
\sigma_x & \tau_{xy} \\
\tau_{xy} & \sigma_y
\end{bmatrix}\begin{bmatrix}
\varepsilon_x & \gamma_{xy}/2 \\
\gamma_{xy}/2 & \varepsilon_y
\end{bmatrix} = \frac{1}{2} (\sigma_x \varepsilon_x + \tau_{xy} \gamma_{xy} + \sigma_y \varepsilon_y)
\]

Using tensorial strain in matrix equations does give the correct \( W \).

• Tensorial Strain Must Be Used In All Matrix Operations
Material Laws
(Constitutive Relations)
(Stress - Strain Equations)

• The stress-strain equations or constitutive relations are a good example of the usefulness of the indicial notation.

• The most basic constitutive relation that we first learn in Mechanics of Materials is the 1-D “Hooke’s Law” equation:
  • \( \sigma = E \varepsilon \), Stress equals Young's modulus times strain.

• Later, we learned that if there is strain in more than one direction, the stress will be a function of all strains.
General 3-D Constitutive Relations

- The first two (of nine) stress equations can be written as shown below. Each of the nine stress components is a function of all nine strain components. Each Q variable is a different function of material properties.

\[
\sigma_{xx} = Q_{xxxx} \varepsilon_{xx} + Q_{xxxx} \varepsilon_{xy} + Q_{xxxx} \varepsilon_{xz} + Q_{xxxx} \varepsilon_{yx} +
\]
\[
Q_{xxxx} \varepsilon_{yy} + Q_{xxxx} \varepsilon_{yz} + Q_{xxxx} \varepsilon_{zx} +
\]
\[
Q_{xxxx} \varepsilon_{zy} + Q_{xxxx} \varepsilon_{zz}
\]

\[
\sigma_{xy} = Q_{xyxx} \varepsilon_{xx} + Q_{xyxx} \varepsilon_{xy} + Q_{xyxx} \varepsilon_{xz} + Q_{xyxx} \varepsilon_{yx} +
\]
\[
Q_{xyxx} \varepsilon_{yy} + Q_{xyxx} \varepsilon_{yz} + Q_{xyxx} \varepsilon_{zx} +
\]
\[
Q_{xyxx} \varepsilon_{zy} + Q_{xyxx} \varepsilon_{zz}
\]
General 3-D Constitutive Relations

- Both the shorthand and full matrix versions of the 3-D constitutive equations are shown below.

\[
\sigma_{ij} = Q_{ijkl} \varepsilon_{kl}
\]

where i, j, k and l represent x, y and z. Sometimes the numbers 1, 2 and 3 are substituted for x, y and z.
Simplifying the Constitutive Relations

• The general constitutive equations have 81 elastic constants. Luckily, the number of constants is reduced for most practical materials.

• Both the stress and strain tensors are symmetric.

\[ \sigma_{ij} = \sigma_{ji} \text{ and } \varepsilon_{ij} = \varepsilon_{ji} \]

This reduces the number of independent stresses and strains to 6.
Constitutive Relations

• Since there are only six independent stresses and strains, another shorthand notation is introduced using numeric subscripts. Let

\[ \begin{align*}
\sigma_1 &= \sigma_{xx} \quad \text{and} \quad \varepsilon_1 = \varepsilon_{xx} \\
\sigma_2 &= \sigma_{yy} \quad \text{and} \quad \varepsilon_2 = \varepsilon_{yy} \\
\sigma_3 &= \sigma_{zz} \quad \text{and} \quad \varepsilon_3 = \varepsilon_{zz} \\
\sigma_4 &= \sigma_{yz} = \sigma_{zy} \quad \text{and} \quad \varepsilon_4 = 2\varepsilon_{yz} = 2\varepsilon_{zy} \\
\sigma_5 &= \sigma_{xz} = \sigma_{zx} \quad \text{and} \quad \varepsilon_5 = 2\varepsilon_{xz} = 2\varepsilon_{zx} \\
\sigma_6 &= \sigma_{xy} = \sigma_{yx} \quad \text{and} \quad \varepsilon_6 = 2\varepsilon_{xy} = 2\varepsilon_{yx}
\end{align*} \]
Constitutive Relations

• This reduces the number of independent material constants to 36.

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\
Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} & Q_{46} \\
Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} & Q_{56} \\
Q_{61} & Q_{62} & Q_{63} & Q_{64} & Q_{65} & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix}
\]

• A thermodynamics proof can be used to show that the Q matrix itself is symmetric, or \( Q_{ij} = Q_{ji} \).

• This reduces the number of independent constants to 21.
Symmetry of Stiffness Matrix

- Symmetry of the stiffness matrix is shown by considering the definition of Strain Energy: \( W = \sigma \varepsilon \)
- The use of Einstein’s Indicial Notation simplifies the proof. The proof is shown here but we will not be covering indicial notation in this course.

\[
\sigma = Q \varepsilon
\]

\[
\sigma_{ij} = Q_{ijkl} \varepsilon_{kl}
\]

\[
W = \sigma \varepsilon = \sigma_{ij} \varepsilon_{ij} = Q_{ijkl} \varepsilon_{kl} \varepsilon_{ij}
\]

These equations can be rewritten with different subscripts:

\[
\sigma_{kl} = Q_{klji} \varepsilon_{ij}
\]

\[
W = \sigma \varepsilon = \sigma_{kl} \varepsilon_{kl} = Q_{klji} \varepsilon_{ij} \varepsilon_{kl}
\]

Both expressions for the strain energy are valid and can be equated:

\[
Q_{ijkl} \varepsilon_{kl} \varepsilon_{ij} = Q_{klji} \varepsilon_{ij} \varepsilon_{kl}
\]

Therefore:

\[
Q_{ijkl} = Q_{klji}
\]

This implies that ‘\(Q\)’, the stiffness matrix is symmetric and reduces the number of independent constants from 36 to 21.
Anisotropy and Material Symmetry

- **General Anisotropy**
  - A material with different material properties in all directions exhibits general anisotropy.
  - 21 independent elastic constants are required to define the stress-strain relationship for this type of material.

- **Orthotropic Material**
  - Has 3 mutually orthogonal planes of elastic symmetry
  - e.g.: A material with the same properties in the +x and -x directions has elastic symmetry about the y-z plane.
  - Most composite materials exhibit elastic symmetry about three planes
  - Requires 9 independent elastic constants
Orthotropic Stress-Strain Equations

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\
Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\
Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{66} \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix}
\]

or \( \sigma_i = Q_{ij} \varepsilon_j \), where \( Q_{ij} \) is know as the stiffness matrix.

Inverting the matrix equation yields:

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix}
\]

or \( \varepsilon_i = S_{ij} \sigma_j \), where \( S_{ij} \) is know as the compliance matrix.

\[
[S] = [Q]^{-1}
\]
Compliance Matrix

• The compliance matrix values are easier to define than the stiffness matrix values.

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix}
= 
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}
\]

\[
S_{11} = \frac{1}{E_{11}} \quad S_{44} = \frac{1}{G_{23}}
\]

\[
S_{22} = \frac{1}{E_{22}} \quad S_{55} = \frac{1}{G_{13}}
\]

\[
S_{33} = \frac{1}{E_{33}} \quad S_{66} = \frac{1}{G_{12}}
\]

\[
S_{12} = S_{21} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}}
\]

\[
S_{13} = S_{31} = -\frac{\nu_{13}}{E_{11}} = -\frac{\nu_{31}}{E_{33}}
\]

\[
S_{23} = S_{32} = -\frac{\nu_{23}}{E_{22}} = -\frac{\nu_{32}}{E_{33}}
\]
Stiffness Matrix

• The $Q_{ij}$ components are found by inverting $S_{ij}$

$$Q_{11} = \frac{E_{11}(1 - \nu_{23}\nu_{32})}{\Delta}$$

$$Q_{22} = \frac{E_{22}(1 - \nu_{31}\nu_{13})}{\Delta}$$

$$Q_{33} = \frac{E_{33}(1 - \nu_{12}\nu_{21})}{\Delta}$$

$$Q_{44} = G_{23}$$

$$Q_{55} = G_{13}$$

$$Q_{66} = G_{12}$$

$$Q_{12} = \frac{E_{11}(\nu_{21} + \nu_{31}\nu_{23})}{\Delta} = \frac{E_{22}(\nu_{12} + \nu_{32}\nu_{13})}{\Delta}$$

$$Q_{13} = \frac{E_{11}(\nu_{31} + \nu_{21}\nu_{32})}{\Delta} = \frac{E_{33}(\nu_{13} + \nu_{12}\nu_{23})}{\Delta}$$

$$Q_{23} = \frac{E_{22}(\nu_{32} + \nu_{12}\nu_{31})}{\Delta} = \frac{E_{33}(\nu_{23} + \nu_{21}\nu_{13})}{\Delta}$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}$$
Isotropic Stress-Strain Relations

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yz}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yz}
\end{bmatrix}
= \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
\]
Isotropic Stress-Strain Relations

(Matrix Form of Equations)
\[ \varepsilon = E^{-1} \sigma \quad \sigma = E \varepsilon \]

(Index Form of Equations)
\[ \sigma_{ij} = \frac{E \nu}{(1+\nu)(1-2\nu)} \delta_{ij} \varepsilon_{kk} + \frac{E}{(1+\nu)} \varepsilon_{ij} \]

or
\[ \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2 \mu \varepsilon_{ij} \]

Where \( \lambda \) and \( \mu \) are known as Lame’s constants.

Note that only 2 independent constants are needed to describe isotropic material behavior.

Also note that \( \mu = G = \) shear stiffness or modulus of rigidity
Common 2-D Conditions: Plane Stress

- Applied to thin flat plates where the loads are generally in the plane of the plate.
- Assume normal and shear stresses in the z-direction are zero:
  \[ \sigma_z = \tau_{xz} = \tau_{yz} = 0 \]
- Assume all other stresses and strains do not vary through the thickness.
- Substitute these assumptions into the general isotropic material law equations to get the following:

\[
\gamma_{yz} = \gamma_{xz} = 0 \quad \text{and} \quad \varepsilon_z = -\frac{\nu}{(1-\nu)}(\varepsilon_x + \varepsilon_y)
\]
Common 2-D Conditions: Plane Strain

• Applied to long structures where loads are in the transverse direction (long pressurized cylinders). Can be applied to other structures where strain is restricted in the z-direction.

• Assume strains along the long axis of the cylinder are zero z-direction):

\[ \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \]

• Substitute these assumptions into the general isotropic material law equations to get the following:

\[ \tau_{yz} = \tau_{xz} = 0 \quad \text{and} \quad \sigma_z = \nu(\sigma_x + \sigma_y) \]
Material Laws for Plane Problems

Plane Stress:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

Plane Strain:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & \frac{1-2\nu}{2}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]
Stress Resultants

It is sometimes convenient to define *stress resultants* \((n_x, n_y, n_{xy})\) as an alternative for stresses. Calculate the stress resultant as force/width (not force/thickness). Units are force/length.

\[
\begin{align*}
    n_x &= \int_{-t/2}^{t/2} \sigma_x \, dz \\
    n_y &= \int_{-t/2}^{t/2} \sigma_y \, dz \\
    n_{xy} &= \int_{-t/2}^{t/2} \tau_{xy} \, dz
\end{align*}
\]

**Plane Stress**
\[
\begin{bmatrix}
    n_x \\
    n_y \\
    n_{xy}
\end{bmatrix}
= \frac{Et}{1-\nu^2}
\begin{bmatrix}
    1 & \nu & 0 \\
    \nu & 1 & 0 \\
    0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\begin{bmatrix}
    \varepsilon_x \\
    \varepsilon_y \\
    \gamma_{xy}
\end{bmatrix}
\]

**Plane Strain**
\[
\begin{bmatrix}
    n_x \\
    n_y \\
    n_{xy}
\end{bmatrix}
= \frac{Et}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
    1-\nu & \nu & 0 \\
    \nu & 1-\nu & 0 \\
    0 & 0 & \frac{1-2\nu}{2}
\end{bmatrix}
\begin{bmatrix}
    \varepsilon_x \\
    \varepsilon_y \\
    \gamma_{xy}
\end{bmatrix}
\]
Total State of Strain

• Thermal strains or initial strains can be added to strains caused by applied loads:

\[ \varepsilon = E^{-1} \sigma + \varepsilon^0 \]

\[ \sigma = E(\varepsilon - \varepsilon^0) \]

• A change in temperature causes the following strains in an isotropic body:

\[ \varepsilon_x^o = \varepsilon_y^o = \varepsilon_z^o = \alpha \Delta T \]

\[ \gamma_{xy}^o = \gamma_{xz}^o = \gamma_{yz}^o = 0 \]

where \( \alpha \) is the coefficient of thermal expansion.
2D Equilibrium Element

State of stress acting on a differential planar element at point $O$ in a 2-D body.

Volumetric (or body) forces are represented by $\bar{p}_v$

The volumetric differential element has dimensions of $dx$, $dy$, and $dz$ into the page.

The $x$-equilibrium equation is shown below:

$$\sum F_x = 0: \left( \sigma_x + \frac{\partial \sigma_x}{\partial x} \right) dy dz + \left( \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \right) dx dz - \sigma_x dy dz - \tau_{yx} dx dz + \bar{p}_{Vx} dx dy dz = 0$$

Dividing by $dx dy dz$ yields:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \bar{p}_{Vx} = 0$$
2D Equilibrium Equations

Writing a force balance equation in the y-direction and a moment balance equation about point \( O \) yields:

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \bar{p}_{Vy} = 0 \quad \tau_{yx} = \tau_{xy}
\]

Writing the 2-D equilibrium equations in matrix form:

\[
\begin{bmatrix}
\partial_x & 0 & \partial_y \\
0 & \partial_y & \partial_x
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
+ \begin{bmatrix}
\bar{p}_{Vx} \\
\bar{p}_{Vy}
\end{bmatrix} = 0
\]

\[
D^T \sigma + \bar{p}_V = 0
\]
Stress Functions

It is often convenient to express the different stresses in terms of a single ‘stress function’. Then, instead of solving for 3 different stresses, there will only be one unknown stress function.

Airy’s stress function ($\psi$) is most common. In the absence of body forces it is:

$$
\sigma_x = \frac{\partial^2 \psi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y}
$$

The stress function must satisfy the equilibrium equations.

Verify that these equations satisfy equilibrium.
Boundary Conditions

Force Boundary Conditions along this edge.

Displacement Boundary Conditions along this edge.

The entire boundary of the structure must be defined as $S_u$ OR $S_p$.

$u = \bar{u}$ on $S_u$

Where $\bar{u}$ are the prescribed displacements.

Force boundary condition may be pressure, moment, point load, or zero load.

Applied surface forces (per unit area) are referred to as surface tractions $\bar{p}$.

$p = \bar{p}$ on $S_p$
Governing Equations

Equilibrium (3):
\[ \frac{\partial \sigma_{ij}}{\partial x_i} + P_{Vi} = 0 \]

Material Law (6):
\[ \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2 \mu \varepsilon_{ij} \]

Strain Displacement Equations (6):
\[ \varepsilon_{ik} = \frac{1}{2} (u_{i,k} + u_{k,i}) \]

15 Equations
with
15 unknowns:
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yz}
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
= 
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix}
\]
Displacement Formulation of Governing Equations

Matrix Form of Equations

Strain Displacement Equations: \( \varepsilon = D \ u \) \hspace{1cm} (1)

Material Law: \( \sigma = E \ \varepsilon \) \hspace{1cm} (2)

Equilibrium: \( D^T \sigma + \bar{p}_V = 0 \) \hspace{1cm} (3)

Substitute (1) into (2) into (3):

\[
D^T E D \ u + \bar{p}_V = 0
\]

\[
\nabla^2 u_x + \frac{1}{1 - 2\nu} \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \frac{\bar{p}_{Vx}}{G} = 0
\]

\[
\nabla^2 u_y + \frac{1}{1 - 2\nu} \frac{\partial}{\partial y} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \frac{\bar{p}_{Vy}}{G} = 0
\]

\[
\nabla^2 u_z + \frac{1}{1 - 2\nu} \frac{\partial}{\partial z} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \frac{\bar{p}_{Vz}}{G} = 0
\]

Now there are only 3 equations and 3 unknowns:

\( u_x, u_y, u_z, \)
Displacement Formulation of Governing Equations

Shorthand Notation

\[ G\nabla^2 u_i + (\lambda + G)u_{k,ki} + \overline{p}_{Vi} = 0 \]

If there are no body forces than this can be simplified even further:

\[ \nabla^2 \nabla^2 u_i = 0 \]

This is the classic biharmonic differential equation that appears frequently in mathematics. \( \nabla^2 \) is the Laplacian or harmonic operator:

\[ \nabla^2 u_i = \partial_x^2 + \partial_y^2 + \partial_z^2 \]
Next Class

• Stress Analysis
• Engineering Beam Theory
• Torsion Theory