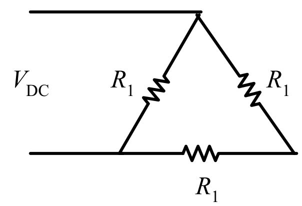
Chapter 7: *Induction Motors*

7-1. A dc test is performed on a 460-V Δ-connected 100-hp induction motor. If V_{DC} = 21 V and I_{DC} = 72 A, what is the stator resistance R_1? *Why is this so?*

SOLUTION If this motor's armature is connected in delta, then there will be two phases in parallel with one phase between the lines tested.



Therefore, the stator resistance R_1 will be

$$\frac{V_{\text{DC}}}{I_{\text{DC}}} = \frac{R_1(R_1 + R_1)}{R_1 + (R_1 + R_1)} = \frac{2}{3} R_1$$

$$R_1 = \frac{3}{2}\frac{V_{\text{DC}}}{I_{\text{DC}}} = \frac{3}{2}\left(\frac{21\text{ V}}{72\text{ A}}\right) = 0.438\ \Omega$$

7-2. A 220-V three-phase six-pole 50-Hz induction motor is running at a slip of 3.5 percent. Find:

(a) The speed of the magnetic fields in revolutions per minute

(b) The speed of the rotor in revolutions per minute

(c) The slip speed of the rotor

(d) The rotor frequency in hertz

SOLUTION

(a) The speed of the magnetic fields is

$$n_{\text{sync}} = \frac{120 f_e}{P} = \frac{120(50\text{ Hz})}{6} = 1000\text{ r/min}$$

(b) The speed of the rotor is

$$n_m = (1-s)\ n_{\text{sync}} = (1-0.035)(1000\text{ r/min}) = 965\text{ r/min}$$

(c) The slip speed of the rotor is

$$n_{\text{slip}} = s n_{\text{sync}} = (0.035)(1000\text{ r/min}) = 35\text{ r/min}$$

(d) The rotor frequency is

$$f_r = \frac{n_{\text{slip}} P}{120} = \frac{(35\text{ r/min})(6)}{120} = 1.75\text{ Hz}$$

7-3. Answer the questions in Problem 7-2 for a 480-V three-phase four-pole 60-Hz induction motor running at a slip of 0.025.

SOLUTION

(a) The speed of the magnetic fields is

$$n_{\text{sync}} = \frac{120 f_e}{P} = \frac{120(60 \text{ Hz})}{4} = 1800 \text{ r/min}$$

(b) The speed of the rotor is

$$n_m = (1-s)\ n_{\text{sync}} = (1-0.025)(1800 \text{ r/min}) = 1755 \text{ r/min}$$

(c) The slip speed of the rotor is

$$n_{\text{slip}} = s n_{\text{sync}} = (0.025)(1800 \text{ r/min}) = 45 \text{ r/min}$$

(d) The rotor frequency is

$$f_r = \frac{n_{\text{slip}} P}{120} = \frac{(45 \text{ r/min})(4)}{120} = 1.5 \text{ Hz}$$

7-4. A three-phase 60-Hz induction motor runs at 715 r/min at no load and at 670 r/min at full load.

(a) How many poles does this motor have?

(b) What is the slip at rated load?

(c) What is the speed at one-quarter of the rated load?

(d) What is the rotor's electrical frequency at one-quarter of the rated load?

SOLUTION

(a) This machine has 10 poles, which produces a synchronous speed of

$$n_{\text{sync}} = \frac{120 f_e}{P} = \frac{120(60 \text{ Hz})}{10} = 720 \text{ r/min}$$

(b) The slip at rated load is

$$s = \frac{n_{\text{sync}} - n_m}{n_{\text{sync}}} \times 100\% = \frac{720-670}{720} \times 100\% = 6.94\%$$

(c) The motor is operating in the linear region of its torque-speed curve, so the slip at ¼ load will be

$$s = 0.25(0.0694) = 0.0174$$

The resulting speed is

$$n_m = (1-s)\ n_{\text{sync}} = (1-0.0174)(720 \text{ r/min}) = 707 \text{ r/min}$$

(d) The electrical frequency at ¼ load is

$$f_r = s f_e = (0.0174)(60 \text{ Hz}) = 1.04 \text{ Hz}$$

7-5. A 50-kW 440-V 50-Hz two-pole induction motor has a slip of 6 percent when operating at full-load conditions. At full-load conditions, the friction and windage losses are 520 W, and the core losses are 500 W. Find the following values for full-load conditions:

(a) The shaft speed n_m

(b) The output power in watts

(c) The load torque τ_{load} in newton-meters

(d) The induced torque τ_{ind} in newton-meters

(e) The rotor frequency in hertz

SOLUTION

(a) The synchronous speed of this machine is

$$n_{\text{sync}} = \frac{120 f_e}{P} = \frac{120(50\text{ Hz})}{2} = 3000\text{ r/min}$$

Therefore, the shaft speed is

$$n_m = (1-s)\ n_{\text{sync}} = (1-0.06)(3000\text{ r/min}) = 2820\text{ r/min}$$

(b) The output power in watts is 50 kW (stated in the problem).

(c) The load torque is

$$\tau_{\text{load}} = \frac{P_{\text{OUT}}}{\omega_m} = \frac{50\text{ kW}}{(2820\text{ r/min})\left(\dfrac{2\pi\text{ rad}}{1\text{ r}}\right)\left(\dfrac{1\text{ min}}{60\text{ s}}\right)} = 169.3\text{ N}\cdot\text{m}$$

(d) The induced torque can be found as follows:

$$P_{\text{conv}} = P_{\text{OUT}} + P_{\text{F\&W}} + P_{\text{core}} + P_{\text{misc}} = 50\text{ kW} + 520\text{ W} + 500\text{ W} = 51.2\text{ kW}$$

$$\tau_{\text{ind}} = \frac{P_{\text{conv}}}{\omega_m} = \frac{51.2\text{ kW}}{(2820\text{ r/min})\left(\dfrac{2\pi\text{ rad}}{1\text{ r}}\right)\left(\dfrac{1\text{ min}}{60\text{ s}}\right)} = 173.4\text{ N}\cdot\text{m}$$

(e) The rotor frequency is

$$f_r = sf_e = (0.06)(50\text{ Hz}) = 3.00\text{ Hz}$$

7-6. A three-phase 60-Hz two-pole induction motor runs at a no-load speed of 3580 r/min and a full-load speed of 3440 r/min. Calculate the slip and the electrical frequency of the rotor at no-load and full-load conditions. What is the speed regulation of this motor [Equation (4-57)]?

SOLUTION The synchronous speed of this machine is 3600 r/min. The slip and electrical frequency at no-load conditions is

$$s_{\text{nl}} = \frac{n_{\text{sync}} - n_{\text{nl}}}{n_{\text{sync}}} \times 100\% = \frac{3600-3580}{3600} \times 100\% = 0.56\%$$

$$f_{r,\text{nl}} = sf_e = (0.0056)(60\text{ Hz}) = 0.33\text{ Hz}$$

The slip and electrical frequency at full load conditions is

$$s_{\text{fl}} = \frac{n_{\text{sync}} - n_{\text{nl}}}{n_{\text{sync}}} \times 100\% = \frac{3600-3440}{3600} \times 100\% = 4.44\%$$

$$f_{r,\text{fl}} = sf_e = (0.0444)(60\text{ Hz}) = 2.67\text{ Hz}$$

The speed regulation is

$$\text{SR} = \frac{n_{\text{nl}} - n_{\text{fl}}}{n_{\text{fl}}} \times 100\% = \frac{3580-3440}{3440} \times 100\% = 4.1\%$$

7-7. A 208-V four-pole 60-Hz Y-connected wound-rotor induction motor is rated at 15 hp. Its equivalent circuit components are

$R_1 = 0.220\ \Omega$ $\quad$ $R_2 = 0.127\ \Omega$ $\quad$ $X_M = 15.0\ \Omega$

$X_1 = 0.430\ \Omega$ $\quad$ $X_2 = 0.430\ \Omega$

$P_{\text{mech}} = 300\text{ W}$ $\quad$ $P_{\text{misc}} \approx 0$ $\quad$ $P_{\text{core}} = 200\text{ W}$

For a slip of 0.05, find

(a) The line current I_L

(b) The stator copper losses P_{SCL}

(c) The air-gap power P_{AG}

(d) The power converted from electrical to mechanical form P_{conv}

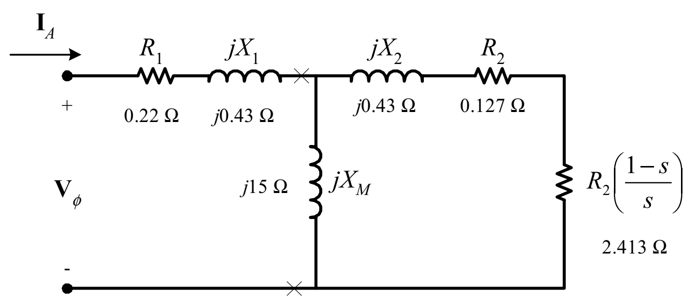
(e) The induced torque τ_{ind}

(f) The load torque τ_{load}

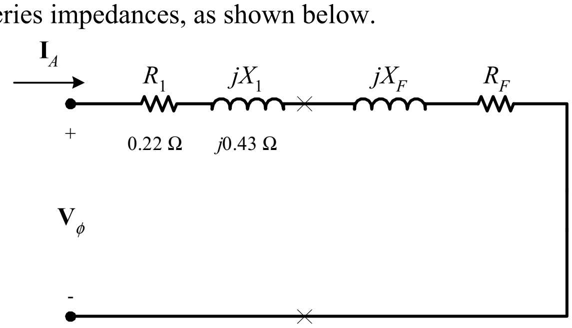
(g) The overall machine efficiency

(h) The motor speed in revolutions per minute and radians per second

SOLUTION The equivalent circuit of this induction motor is shown below:



(a) The easiest way to find the line current (or armature current) is to get the equivalent impedance Z_F of the rotor circuit in parallel with jX_M, and then calculate the current as the phase voltage divided by the sum of the series impedances, as shown below.



The equivalent impedance of the rotor circuit in parallel with jX_M is:

$$Z_F = \frac{1}{\dfrac{1}{jX_M} + \dfrac{1}{Z_2}} = \frac{1}{\dfrac{1}{j15\,\Omega} + \dfrac{1}{2.54 + j0.43}} = 2.337 + j0.803 = 2.47\angle 19^\circ\ \Omega$$

The phase voltage is $208/\sqrt{3} = 120$ V, so line current I_L is

$$I_L = I_A = \frac{V_\phi}{R_1 + jX_1 + R_F + jX_F} = \frac{120\angle 0° \text{ V}}{0.22\,\Omega + j0.43\,\Omega + 2.337\,\Omega + j0.803\,\Omega}$$

$$I_L = I_A = 42.3\angle -25.7° \text{ A}$$

(b) The stator copper losses are

$$P_{\text{SCL}} = 3I_A^{\;2}R_1 = 3(42.3 \text{ A})^2(0.22\ \Omega) = 1180 \text{ W}$$

(c) The air gap power is $P_{\text{AG}} = 3I_2^{\;2}\frac{R_2}{s} = 3I_A^{\;2}R_F$

(Note that $3I_A^{\;2}R_F$ is equal to $3I_2^{\;2}\frac{R_2}{s}$, since the only resistance in the original rotor circuit was R_2/s, and the resistance in the Thevenin equivalent circuit is R_F. The power consumed by the Thevenin equivalent circuit must be the same as the power consumed by the original circuit.)

$$P_{\text{AG}} = 3I_2^{\;2}\frac{R_2}{s} = 3I_A^{\;2}R_F = 3(42.3 \text{ A})^2(2.337\ \Omega) = 12.54 \text{ kW}$$

(d) The power converted from electrical to mechanical form is

$$P_{\text{conv}} = (1-s)P_{\text{AG}} = (1-0.05)(12.54 \text{ kW}) = 11.92 \text{ kW}$$

(e) The induced torque in the motor is

$$\tau_{\text{ind}} = \frac{P_{\text{AG}}}{\omega_{\text{sync}}} = \frac{12.54 \text{ kW}}{(1800 \text{ r/min})\left(\frac{2\pi \text{ rad}}{1 \text{ r}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right)} = 66.5 \text{ N}\cdot\text{m}$$

(f) The output power of this motor is

$$P_{\text{OUT}} = P_{\text{conv}} - P_{\text{mech}} - P_{\text{core}} - P_{\text{misc}} = 11.92 \text{ kW} - 300 \text{ W} - 200 \text{ W} - 0 \text{ W} = 11.42 \text{ kW}$$

The output speed is

$$n_m = (1-s)\,n_{\text{sync}} = (1-0.05)(1800 \text{ r/min}) = 1710 \text{ r/min}$$

Therefore the load torque is

$$\tau_{\text{load}} = \frac{P_{\text{OUT}}}{\omega_m} = \frac{11.42 \text{ kW}}{(1710 \text{ r/min})\left(\frac{2\pi \text{ rad}}{1 \text{ r}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right)} = 63.8 \text{ N}\cdot\text{m}$$

(g) The overall efficiency is

$$\eta = \frac{P_{\text{OUT}}}{P_{\text{IN}}} \times 100\% = \frac{P_{\text{OUT}}}{3V_\phi I_A \cos\theta} \times 100\%$$

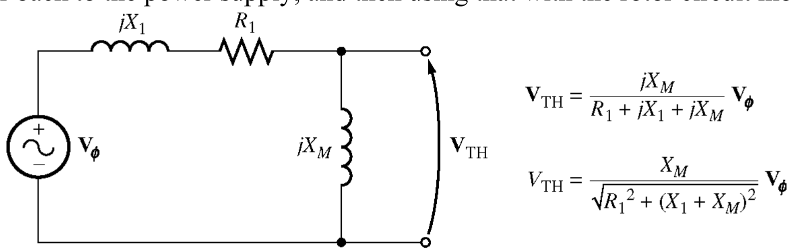
$$\eta = \frac{11.42 \text{ kW}}{3(120 \text{ V})(42.3 \text{ A})\cos 25.7°} \times 100\% = 83.2\%$$

(h) The motor speed in revolutions per minute is 1710 r/min. The motor speed in radians per second is

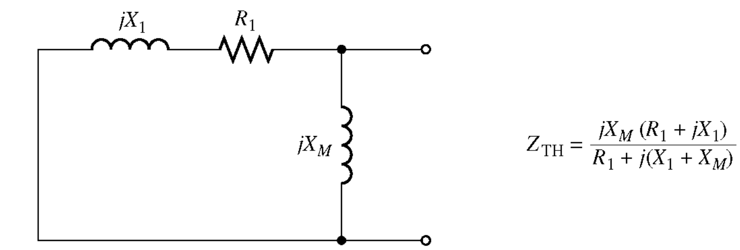
$$\omega_m = (1710 \text{ r/min})\left(\frac{2\pi \text{ rad}}{1 \text{ r}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 179 \text{ rad/s}$$

7-8. For the motor in Problem 7-7, what is the slip at the pullout torque? What is the pullout torque of this motor?

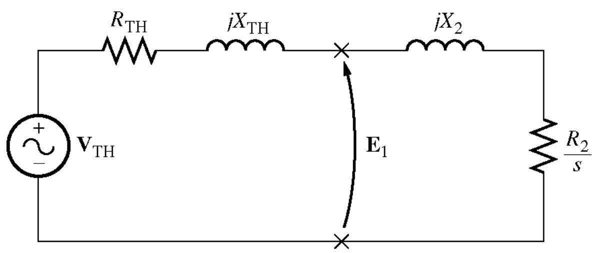
Solution The slip at pullout torque is found by calculating the Thevenin equivalent of the input circuit from the rotor back to the power supply, and then using that with the rotor circuit model.



(*a*)



(*b*)



(*c*)

$$Z_{TH} = \frac{jX_M(R_1 + jX_1)}{R_1 + j(X_1 + X_M)} = \frac{(j15\ \Omega)(0.22\ \Omega + j0.43\ \Omega)}{0.22\ \Omega + j(0.43\ \Omega + 15\ \Omega)} = 0.208 + j0.421\ \Omega = 0.470\angle 63.7°\ \Omega$$

$$\mathbf{V}_{TH} = \frac{jX_M}{R_1 + j(X_1 + X_M)}\mathbf{V}_\phi = \frac{(j15\,\Omega)}{0.22\,\Omega + j(0.43\,\Omega + 15\,\Omega)}(120\angle 0°\text{ V}) = 116.7\angle 0.8°\text{ V}$$

The slip at pullout torque is

$$s_{max} = \frac{R_2}{\sqrt{R_{TH}^2 + (X_{TH} + X_2)^2}}$$

$$s_{max} = \frac{0.127\ \Omega}{\sqrt{(0.208\ \Omega)^2 + (0.421\ \Omega + 0.430\ \Omega)^2}} = 0.145$$

The pullout torque of the motor is

$$\tau_{\max} = \frac{3V_{\text{TH}}^2}{2\omega_{\text{sync}}\left[R_{\text{TH}} + \sqrt{R_{\text{TH}}^2 + \left(X_{\text{TH}} + X_2\right)^2}\right]}$$

$$\tau_{\max} = \frac{3(116.7\text{ V})^2}{2(188.5\text{ rad/s})\left[0.208\ \Omega + \sqrt{(0.208\ \Omega)^2 + (0.421\ \Omega\ + 0.430\ \Omega)^2}\right]}$$

$$\tau_{\max} = 100\text{ N}\cdot\text{m}$$

7-9. *(a)* Calculate and plot the torque-speed characteristic of the motor in Problem 7-7. *(b)* Calculate and plot the output power versus speed curve of the motor in Problem 7-7.

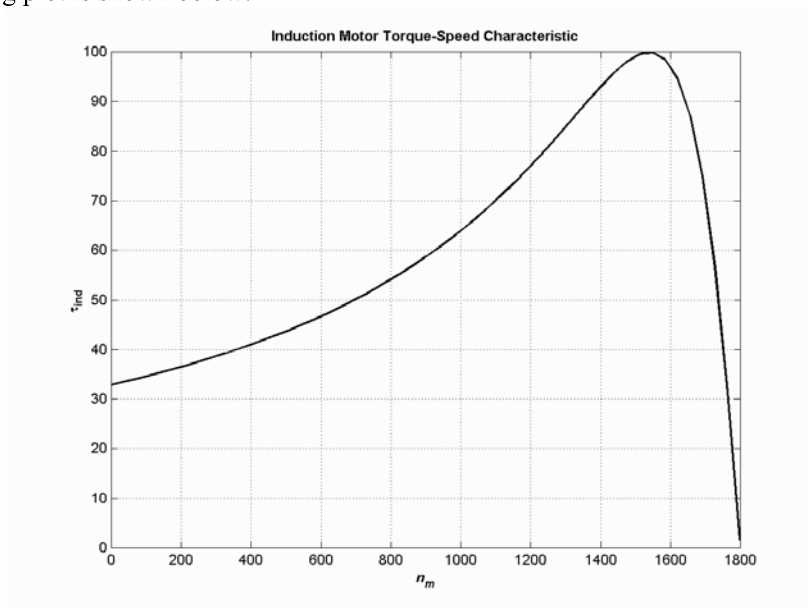
SOLUTION

(a) A MATLAB program to calculate the torque-speed characteristic is shown below.

```
% M-file: prob7_9a.m
% M-file create a plot of the torque-speed curve of the
%   induction motor of Problem 7-7.

% First, initialize the values needed in this program.
r1 = 0.220;                   % Stator resistance
x1 = 0.430;                   % Stator reactance
r2 = 0.127;                   % Rotor resistance
x2 = 0.430;                   % Rotor reactance
xm = 15.0;                    % Magnetization branch reactance
v_phase = 208 / sqrt(3);      % Phase voltage
n_sync = 1800;                % Synchronous speed (r/min)
w_sync = 188.5;               % Synchronous speed (rad/s)

% Calculate the Thevenin voltage and impedance from Equations
% 7-38 and 7-41.
v_th = v_phase * ( xm / sqrt(r1^2 + (x1 + xm)^2) );
z_th = ((j*xm) * (r1 + j*x1)) / (r1 + j*(x1 + xm));
r_th = real(z_th);
x_th = imag(z_th);

% Now calculate the torque-speed characteristic for many
% slips between 0 and 1.  Note that the first slip value
% is set to 0.001 instead of exactly 0 to avoid divide-
% by-zero problems.
s = (0:1:50) / 50;             % Slip
s(1) = 0.001;
nm = (1 - s) * n_sync;         % Mechanical speed

% Calculate torque versus speed
for ii = 1:51
   t_ind(ii) = (3 * v_th^2 * r2 / s(ii)) / ...
           (w_sync * ((r_th + r2/s(ii))^2 + (x_th + x2)^2) );
end

% Plot the torque-speed curve
figure(1);
plot(nm,t_ind,'k-','LineWidth',2.0);
xlabel('\bf\itn_{m}');
ylabel('\bf\tau_{ind}');
```

```
title ('\bfInduction Motor Torque-Speed Characteristic');
grid on;
```

The resulting plot is shown below:

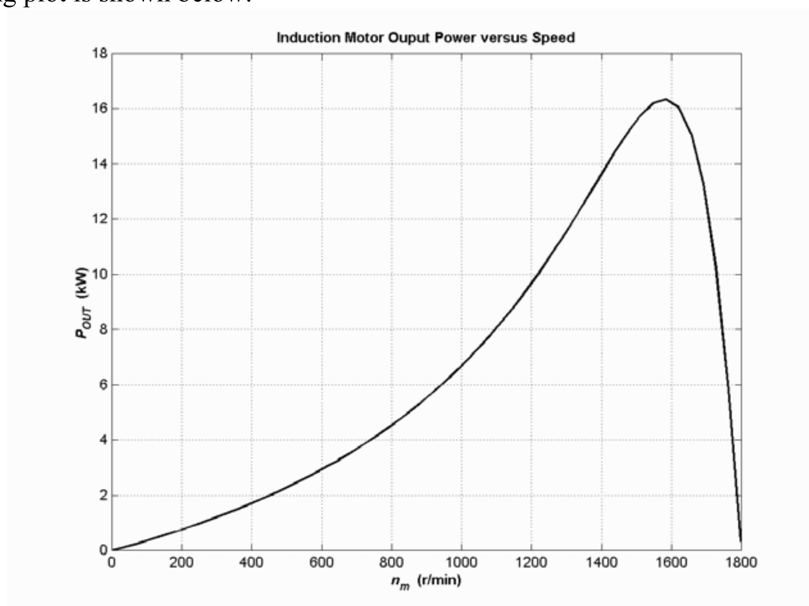


(b) A MATLAB program to calculate the output-power versus speed curve is shown below.

```
% M-file: prob7_9b.m
% M-file create a plot of the output pwer versus speed
%   curve of the induction motor of Problem 7-7.

% First, initialize the values needed in this program.
r1 = 0.220;                 % Stator resistance
x1 = 0.430;                 % Stator reactance
r2 = 0.127;                 % Rotor resistance
x2 = 0.430;                 % Rotor reactance
xm = 15.0;                  % Magnetization branch reactance
v_phase = 208 / sqrt(3);    % Phase voltage
n_sync = 1800;              % Synchronous speed (r/min)
w_sync = 188.5;             % Synchronous speed (rad/s)

% Calculate the Thevenin voltage and impedance from Equations
% 7-38 and 7-41.
v_th = v_phase * ( xm / sqrt(r1^2 + (x1 + xm)^2) );
z_th = ((j*xm) * (r1 + j*x1)) / (r1 + j*(x1 + xm));
r_th = real(z_th);
x_th = imag(z_th);

% Now calculate the torque-speed characteristic for many
% slips between 0 and 1.  Note that the first slip value
% is set to 0.001 instead of exactly 0 to avoid divide-
% by-zero problems.
s = (0:1:50) / 50;           % Slip
s(1) = 0.001;
nm = (1 - s) * n_sync;       % Mechanical speed (r/min)
wm = (1 - s) * w_sync;       % Mechanical speed (rad/s)
```

```
% Calculate torque and output power versus speed
for ii = 1:51
   t_ind(ii) = (3 * v_th^2 * r2 / s(ii)) / ...
           (w_sync * ((r_th + r2/s(ii))^2 + (x_th + x2)^2) );
   p_out(ii) = t_ind(ii) * wm(ii);
end

% Plot the torque-speed curve
figure(1);
plot(nm,p_out/1000,'k-','LineWidth',2.0);
xlabel('\bf\itn_{m}  \rm\bf(r/min)');
ylabel('\bf\itP_{OUT}  \rm\bf(kW)');
title ('\bfInduction Motor Ouput Power versus Speed');
grid on;
```

The resulting plot is shown below:



7-10. For the motor of Problem 7-7, how much additional resistance (referred to the stator circuit) would it be necessary to add to the rotor circuit to make the maximum torque occur at starting conditions (when the shaft is not moving)? Plot the torque-speed characteristic of this motor with the additional resistance inserted.

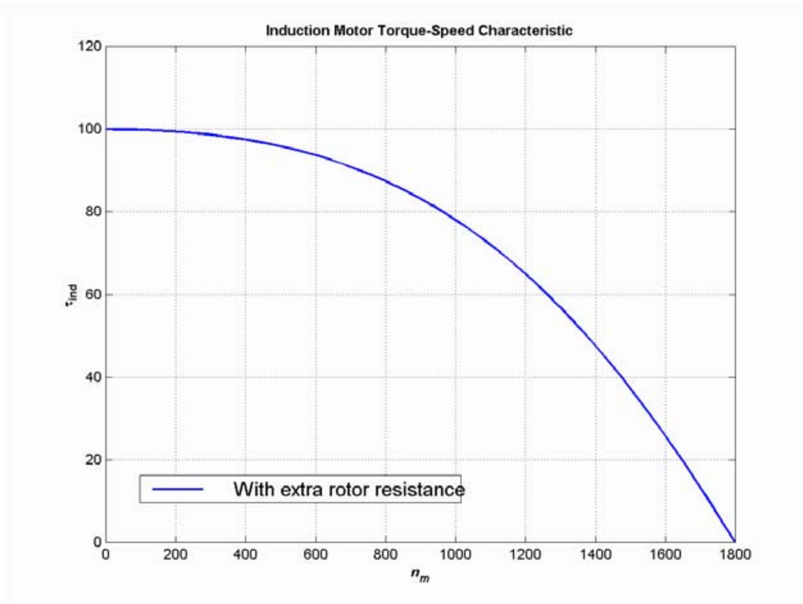
SOLUTION To get the maximum torque at starting, the $s_{\max}$ must be 1.00. Therefore,

$$s_{\max} = \frac{R_2}{\sqrt{R_{\mathrm{TH}}{}^2 + \left(X_{\mathrm{TH}} + X_2\right)^2}}$$

$$1.00 = \frac{R_2}{\sqrt{(0.208\ \Omega)^2 + (0.421\ \Omega\ + 0.430\ \Omega)^2}}$$

$$R_2 = 0.876\,\Omega$$

Therefore, an additional 0.749 Ω must be added to the rotor circuit. The resulting torque-speed characteristic is:

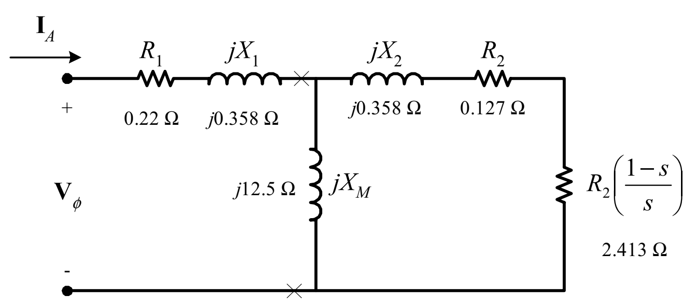


7-11. If the motor in Problem 7-7 is to be operated on a 50-Hz power system, what must be done to its supply voltage? Why? What will the equivalent circuit component values be at 50 Hz? Answer the questions in Problem 7-7 for operation at 50 Hz with a slip of 0.05 and the proper voltage for this machine.

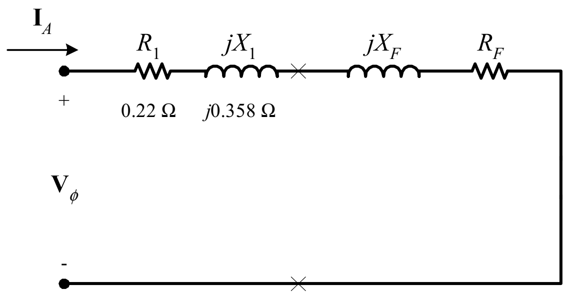
SOLUTION If the input frequency is decreased to 50 Hz, then the applied voltage must be decreased by 5/6 also. If this were not done, the flux in the motor would go into saturation, since

$$\phi = \frac{1}{N} \int_T v \; dt$$

and the period T would be increased. At 50 Hz, the resistances will be unchanged, but the reactances will be reduced to 5/6 of their previous values. The equivalent circuit of the induction motor at 50 Hz is shown below:



(a) The easiest way to find the line current (or armature current) is to get the equivalent impedance Z_F of the rotor circuit in parallel with jX_M, and then calculate the current as the phase voltage divided by the sum of the series impedances, as shown below.



The equivalent impedance of the rotor circuit in parallel with jX_M is:

$$Z_F = \frac{1}{\dfrac{1}{jX_M} + \dfrac{1}{Z_2}} = \frac{1}{\dfrac{1}{j12.5\,\Omega} + \dfrac{1}{2.54 + j0.358}} = 2.310 + j0.804 = 2.45\angle 19.2^\circ\,\Omega$$

The line voltage must be derated by 5/6, so the new line voltage is $V_T = 173.3\text{ V}$. The phase voltage is $173.3/\sqrt{3} = 100$ V, so line current I_L is

$$I_L = I_A = \frac{V_\phi}{R_1 + jX_1 + R_F + jX_F} = \frac{100\angle 0^\circ\text{ V}}{0.22\,\Omega + j0.358\,\Omega + 2.310\,\Omega + j0.804\,\Omega}$$

$$I_L = I_A = 35.9\angle -24.7^\circ\text{ A}$$

(b) The stator copper losses are

$$P_{\text{SCL}} = 3I_A^{\,2}R_1 = 3(35.9\text{ A})^2(0.22\,\Omega) = 851\text{ W}$$

(c) The air gap power is $P_{\text{AG}} = 3I_2^{\,2}\dfrac{R_2}{s} = 3I_A^{\,2}R_F$

(Note that $3I_A^{\,2}R_F$ is equal to $3I_2^{\,2}\dfrac{R_2}{s}$, since the only resistance in the original rotor circuit was R_2/s, and the resistance in the Thevenin equivalent circuit is R_F. The power consumed by the Thevenin equivalent circuit must be the same as the power consumed by the original circuit.)

$$P_{\text{AG}} = 3I_2^{\,2}\frac{R_2}{s} = 3I_A^{\,2}R_F = 3(35.9\text{ A})^2(2.310\,\Omega) = 8.93\text{ kW}$$

(d) The power converted from electrical to mechanical form is

$$P_{\text{conv}} = (1-s)P_{\text{AG}} = (1-0.05)(8.93\text{ kW}) = 8.48\text{ kW}$$

(e) The induced torque in the motor is

$$\tau_{\text{ind}} = \frac{P_{\text{AG}}}{\omega_{\text{sync}}} = \frac{8.48\text{ kW}}{(1500\text{ r/min})\left(\dfrac{2\pi\text{ rad}}{1\text{ r}}\right)\left(\dfrac{1\text{ min}}{60\text{ s}}\right)} = 54.0\text{ N}\cdot\text{m}$$

(f) In the absence of better information, we will treat the mechanical and core losses as constant despite the change in speed. This is not true, but we don't have reason for a better guess. Therefore, the output power of this motor is

$$P_{\text{OUT}} = P_{\text{conv}} - P_{\text{mech}} - P_{\text{core}} - P_{\text{misc}} = 8.48\text{ kW} - 300\text{ W} - 200\text{ W} - 0\text{ W} = 7.98\text{ kW}$$

The output speed is

$$n_m = (1-s)\ n_{\text{sync}} = (1-0.05)(1500 \text{ r/min}) = 1425 \text{ r/min}$$

Therefore the load torque is

$$\tau_{\text{load}} = \frac{P_{\text{OUT}}}{\omega_m} = \frac{7.98 \text{ kW}}{(1425 \text{ r/min})\left(\dfrac{2\pi \text{ rad}}{1 \text{ r}}\right)\left(\dfrac{1 \text{ min}}{60 \text{ s}}\right)} = 53.5 \text{ N}\cdot\text{m}$$

(g) The overall efficiency is

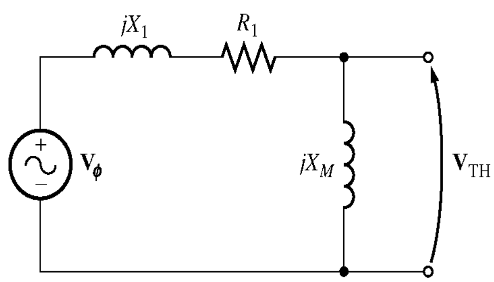
$$\eta = \frac{P_{\text{OUT}}}{P_{\text{IN}}} \times 100\% = \frac{P_{\text{OUT}}}{3V_\phi I_A \cos\theta} \times 100\%$$

$$\eta = \frac{7.98 \text{ kW}}{3(100 \text{ V})(35.9 \text{ A})\cos 24.7°} \times 100\% = 81.6\%$$

(h) The motor speed in revolutions per minute is 1425 r/min. The motor speed in radians per second is

$$\omega_m = (1425 \text{ r/min})\left(\frac{2\pi \text{ rad}}{1 \text{ r}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 149.2 \text{ rad/s}$$

7-12. Figure 7-16*a* shows a simple circuit consisting of a voltage source, a resistor, and two reactances. Find the Thevenin equivalent voltage and impedance of this circuit at the terminals. Then derive the expressions for the magnitude of $\mathbf{V}_{\text{TH}}$ and for R_{TH} given in Equations (7-38) and (7-42).



(*a*)

SOLUTION The Thevenin voltage of this circuit is

$$\mathbf{V}_{\text{TH}} = \frac{jX_M}{R_1 + j(X_1 + X_M)} \mathbf{V}_\phi$$

The *magnitude* of this voltage is

$$V_{\text{TH}} = \frac{X_M}{\sqrt{R_1^{\,2} + (X_1 + X_M)^2}} V_\phi$$

If $X_M >> X_1$, then $R_1^{\,2} + (X_1 + X_M)^2 \approx (X_1 + X_M)^2$, so

$$\boxed{V_{\text{TH}} \approx \frac{X_M}{X_1 + X_M} V_\phi}$$

The Thevenin impedance of this circuit is

$$Z_{\mathrm{TH}} = \frac{jX_M\left(R_1 + jX_1\right)}{R_1 + j\left(X_1 + X_M\right)}$$

$$Z_{\mathrm{TH}} = \frac{jX_M\left(R_1 + jX_1\right)\left[R_1 - j\left(X_1 + X_M\right)\right]}{\left[R_1 + j\left(X_1 + X_M\right)\right]\left[R_1 - j\left(X_1 + X_M\right)\right]}$$

$$Z_{\mathrm{TH}} = \frac{\left[-R_1X_1X_M + R_1X_1X_M + R_1X_M{}^2\right] + j\left[R_1{}^2X_M + X_1{}^2X_M + X_1X_M{}^2\right]}{R_1{}^2 + \left(X_1 + X_M\right)^2}$$

$$Z_{\mathrm{TH}} = R_{\mathrm{TH}} + jX_{\mathrm{TH}} = \frac{R_1X_M{}^2}{R_1{}^2 + \left(X_1 + X_M\right)^2} + j\,\frac{R_1{}^2X_M + X_1{}^2X_M + X_1X_M{}^2}{R_1{}^2 + \left(X_1 + X_M\right)^2}$$

The Thevenin resistance is $R_{\mathrm{TH}} = \dfrac{R_1X_M{}^2}{R_1{}^2 + \left(X_1 + X_M\right)^2}$. If $X_M >> R_1$, then $R_1{}^2 + \left(X_1 + X_M\right)^2 \approx \left(X_1 + X_M\right)^2$, so

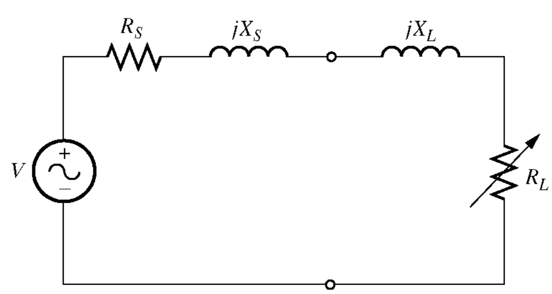
$$\boxed{R_{\mathrm{TH}} \approx R_1\left(\frac{X_M}{X_1 + X_M}\right)^2}$$

The Thevenin reactance is $X_{\mathrm{TH}} = \dfrac{R_1{}^2X_M + X_1{}^2X_M + X_1X_M{}^2}{R_1{}^2 + \left(X_1 + X_M\right)^2}$.

If $X_M >> R_1$ and $X_M >> X_1$ then $X_1X_M{}^2 >> R_1{}^2X_M + X_1{}^2X_M$ and $\left(X_1 + X_M\right)^2 \approx X_M{}^2 >> R_1{}^2$, so

$$\boxed{X_{\mathrm{TH}} \approx \frac{X_1X_M{}^2}{X_M{}^2} = X_1}$$

7-13. Figure P7-1 shows a simple circuit consisting of a voltage source, two resistors, and two reactances in series with each other. If the resistor R_L is allowed to vary but all the other components are constant, at what value of R_L will the maximum possible power be supplied to it? *Prove* your answer. (*Hint:* Derive an expression for load power in terms of *V*, R_S, X_S, R_L and X_L and take the partial derivative of that expression with respect to R_L.) Use this result to derive the expression for the pullout torque [Equation (7-52)].



SOLUTION The current flowing in this circuit is given by the equation

$$\mathbf{I}_L = \frac{\mathbf{V}}{R_S + jX_S + R_L + jX_L}$$

$$I_L = \frac{V}{\sqrt{(R_S + R_L)^2 + (X_S + X_L)^2}}$$

The power supplied to the load is

$$P = I_L^{\,2} R_L = \frac{V^2 R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{\left[(R_S + R_L)^2 + (X_S + X_L)^2\right] V^2 - V^2 R_L \left[2(R_S + R_L)\right]}{\left[(R_S + R_L)^2 + (X_S + X_L)^2\right]^2}$$

To find the point of maximum power supplied to the load, set $\partial P / \partial R_L = 0$ and solve for R_L.

$$\left[(R_S + R_L)^2 + (X_S + X_L)^2\right] V^2 - V^2 R_L \left[2(R_S + R_L)\right] = 0$$

$$\left[(R_S + R_L)^2 + (X_S + X_L)^2\right] = 2R_L (R_S + R_L)$$

$$R_S^{\,2} + 2R_S R_L + R_L^{\,2} + (X_S + X_L)^2 = 2R_S R_L + 2R_L^{\,2}$$

$$R_S^{\,2} + R_L^{\,2} + (X_S + X_L)^2 = 2R_L^{\,2}$$

$$R_S^{\,2} + (X_S + X_L)^2 = R_L^{\,2}$$

Therefore, for maximum power transfer, the load resistor should be

$$\boxed{R_L = \sqrt{R_S^{\,2} + (X_S + X_L)^2}}$$

7-14. A 440-V 50-Hz six-pole Y-connected induction motor is rated at 75 kW. The equivalent circuit parameters are

$R_1 = 0.082\ \Omega$	$R_2 = 0.070\ \Omega$	$X_M = 7.2\ \Omega$
$X_1 = 0.19\ \Omega$	$X_2 = 0.18\ \Omega$	
$P_{\text{F\&W}} = 1.3$ kW	$P_{\text{misc}} = 150$ W	$P_{\text{core}} = 1.4$ kW

For a slip of 0.04, find

(a) The line current I_L

(b) The stator power factor

(c) The rotor power factor

(d) The stator copper losses P_{SCL}

(e) The air-gap power P_{AG}

(f) The power converted from electrical to mechanical form P_{conv}

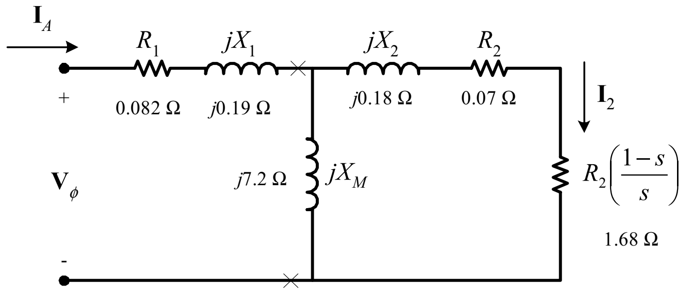
(g) The induced torque τ_{ind}

(h) The load torque τ_{load}

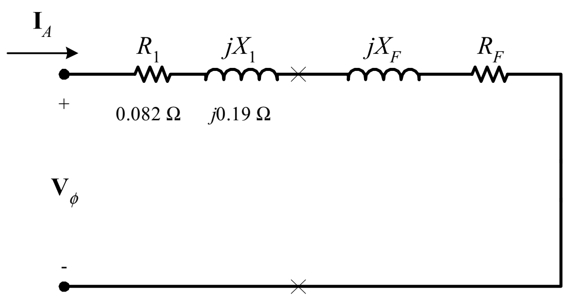
(i) The overall machine efficiency η

(j) The motor speed in revolutions per minute and radians per second

SOLUTION The equivalent circuit of this induction motor is shown below:



(a) The easiest way to find the line current (or armature current) is to get the equivalent impedance Z_F of the rotor circuit in parallel with jX_M, and then calculate the current as the phase voltage divided by the sum of the series impedances, as shown below.



The equivalent impedance of the rotor circuit in parallel with jX_M is:

$$Z_F = \frac{1}{\dfrac{1}{jX_M} + \dfrac{1}{Z_2}} = \frac{1}{\dfrac{1}{j7.2\,\Omega} + \dfrac{1}{1.75 + j0.18}} = 1.557 + j0.550 = 1.67\angle 19.2^\circ\ \Omega$$

The phase voltage is 440/$\sqrt{3}$ = 254 V, so line current I_L is

$$I_L = I_A = \frac{V_\phi}{R_1 + jX_1 + R_F + jX_F} = \frac{254\angle 0^\circ\ \text{V}}{0.082\,\Omega + j0.19\,\Omega + 1.557\,\Omega + j0.550\,\Omega}$$

$$I_L = I_A = 141\angle -24.3^\circ\ \text{A}$$

(b) The stator power factor is

$$\text{PF} = \cos 24.3^\circ = 0.911\ \text{lagging}$$

(c) To find the rotor power factor, we must find the impedance angle of the rotor

$$\theta_R = \tan^{-1}\frac{X_2}{R_2/s} = \tan^{-1}\frac{0.18}{1.75} = 5.87^\circ$$

Therefore the rotor power factor is

$$\text{PF}_R = \cos 5.87^\circ = 0.995\ \text{lagging}$$

(d) The stator copper losses are

$$P_{\text{SCL}} = 3I_A^{\,2}R_1 = 3(141\ \text{A})^2(0.082\ \Omega) = 4890\ \text{W}$$

(e) The air gap power is $P_{\text{AG}} = 3I_2^{\,2}\dfrac{R_2}{s} = 3I_A^{\,2}R_F$

(Note that $3I_A^{\,2}R_F$ is equal to $3I_2^{\,2}\dfrac{R_2}{s}$, since the only resistance in the original rotor circuit was R_2/s, and the resistance in the Thevenin equivalent circuit is R_F. The power consumed by the Thevenin equivalent circuit must be the same as the power consumed by the original circuit.)

$$P_{\text{AG}} = 3I_2^{\,2}\frac{R_2}{s} = 3I_A^{\,2}R_F = 3(141\text{ A})^2(1.557\ \Omega) = 92.6\text{ kW}$$

(f) The power converted from electrical to mechanical form is

$$P_{\text{conv}} = (1-s)P_{\text{AG}} = (1-0.04)(92.6\text{ kW}) = 88.9\text{ kW}$$

(g) The synchronous speed of this motor is

$$n_{\text{sync}} = \frac{120f_e}{P} = \frac{120(50\text{ Hz})}{6} = 1000\text{ r/min}$$

$$\omega_{\text{sync}} = (1000\text{ r/min})\left(\frac{2\pi\text{ rad}}{1\text{ r}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 104.7\text{ rad/s}$$

Therefore the induced torque in the motor is

$$\tau_{\text{ind}} = \frac{P_{\text{AG}}}{\omega_{\text{sync}}} = \frac{92.6\text{ kW}}{(1000\text{ r/min})\left(\frac{2\pi\text{ rad}}{1\text{ r}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right)} = 884\text{ N}\cdot\text{m}$$

(h) The output power of this motor is

$$P_{\text{OUT}} = P_{\text{conv}} - P_{\text{mech}} - P_{\text{core}} - P_{\text{misc}} = 88.9\text{ kW} - 1.3\text{ kW} - 1.4\text{ kW} - 300\text{ W} = 85.9\text{ kW}$$

The output speed is

$$n_m = (1-s)n_{\text{sync}} = (1-0.04)(1000\text{ r/min}) = 960\text{ r/min}$$

Therefore the load torque is

$$\tau_{\text{load}} = \frac{P_{\text{OUT}}}{\omega_m} = \frac{85.9\text{ kW}}{(960\text{ r/min})\left(\frac{2\pi\text{ rad}}{1\text{ r}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right)} = 854\text{ N}\cdot\text{m}$$

(i) The overall efficiency is

$$\eta = \frac{P_{\text{OUT}}}{P_{\text{IN}}}\times 100\% = \frac{P_{\text{OUT}}}{3V_\phi I_A\cos\theta}\times 100\%$$

$$\eta = \frac{85.9\text{ kW}}{3(254\text{ V})(141\text{ A})\cos 24.3^\circ}\times 100\% = 87.7\%$$

(j) The motor speed in revolutions per minute is 960 r/min. The motor speed in radians per second is

$$\omega_m = (960\text{ r/min})\left(\frac{2\pi\text{ rad}}{1\text{ r}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 100.5\text{ rad/s}$$

7-15. For the motor in Problem 7-14, what is the pullout torque? What is the slip at the pullout torque? What is the rotor speed at the pullout torque?

SOLUTION The slip at pullout torque is found by calculating the Thevenin equivalent of the input circuit from the rotor back to the power supply, and then using that with the rotor circuit model.

$$Z_{TH} = \frac{jX_M(R_1 + jX_1)}{R_1 + j(X_1 + X_M)} = \frac{(j7.2\ \Omega)(0.082\ \Omega + j0.19\ \Omega)}{0.082\ \Omega + j(0.19\ \Omega + 7.2\ \Omega)} = 0.0778 + j0.1860\ \Omega = 0.202\angle 67.3°\ \Omega$$

$$\mathbf{V}_{TH} = \frac{jX_M}{R_1 + j(X_1 + X_M)}\mathbf{V}_\phi = \frac{(j7.2\ \Omega)}{0.082\ \Omega + j(0.19\ \Omega + 7.2\ \Omega)}(254\angle 0°\ \text{V}) = 247.5\angle 0.6°\ \text{V}$$

The slip at pullout torque is

$$s_{max} = \frac{R_2}{\sqrt{R_{TH}^{\ 2} + (X_{TH} + X_2)^2}}$$

$$s_{max} = \frac{0.070\ \Omega}{\sqrt{(0.0778\ \Omega)^2 + (0.186\ \Omega + 0.180\ \Omega)^2}} = 0.187$$

The pullout torque of the motor is

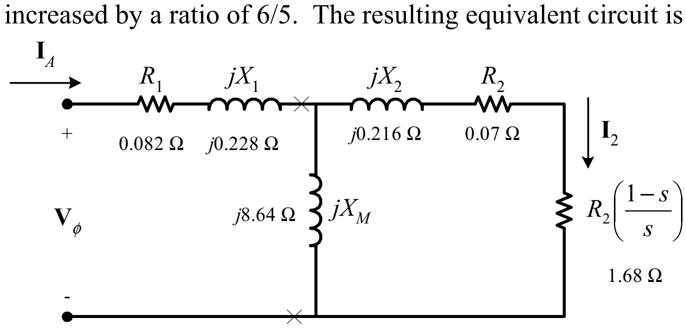
$$\tau_{max} = \frac{3V_{TH}^2}{2\omega_{sync}\left[R_{TH} + \sqrt{R_{TH}^2 + (X_{TH} + X_2)^2}\right]}$$

$$\tau_{max} = \frac{3(247.5\ \text{V})^2}{2(104.7\ \text{rad/s})\left[0.0778\ \Omega + \sqrt{(0.0778\ \Omega)^2 + (0.186\ \Omega + 0.180\ \Omega)^2}\right]}$$

$$\tau_{max} = 1941\ \text{N}\cdot\text{m}$$

7-16. If the motor in Problem 7-14 is to be driven from a 440-V 60-Hz power supply, what will the pullout torque be? What will the slip be at pullout?

SOLUTION If this motor is driven from a 60 Hz source, the resistances will be unchanged and the reactances will be increased by a ratio of 6/5. The resulting equivalent circuit is shown below.



The slip at pullout torque is found by calculating the Thevenin equivalent of the input circuit from the rotor back to the power supply, and then using that with the rotor circuit model.

$$Z_{TH} = \frac{jX_M(R_1 + jX_1)}{R_1 + j(X_1 + X_M)} = \frac{(j8.64\ \Omega)(0.082\ \Omega\ + j0.228\ \Omega)}{0.082\ \Omega\ + j(0.228\ \Omega + 8.64\ \Omega)} = 0.0778 + j0.223\ \Omega = 0.236\angle 70.7°\ \Omega$$

$$\mathbf{V}_{TH} = \frac{jX_M}{R_1 + j(X_1 + X_M)}\mathbf{V}_\phi = \frac{j8.64\ \Omega}{0.082\ \Omega\ + j(0.228\ \Omega + 8.64\ \Omega)}(254\angle 0°\ \text{V}) = 247.5\angle 0.5°\ \text{V}$$

The slip at pullout torque is

$$s_{\max} = \frac{R_2}{\sqrt{R_{\mathrm{TH}}{}^2 + \left(X_{\mathrm{TH}} + X_2\right)^2}}$$

$$s_{\max} = \frac{0.070\,\Omega}{\sqrt{(0.0778\,\Omega)^2 + (0.223\,\Omega + 0.216\,\Omega)^2}} = 0.157$$

The synchronous speed of this motor is

$$n_{\mathrm{sync}} = \frac{120 f_e}{P} = \frac{120(60\ \mathrm{Hz})}{6} = 1200\ \mathrm{r/min}$$

$$\omega_{\mathrm{sync}} = (1200\ \mathrm{r/min})\left(\frac{2\pi\ \mathrm{rad}}{1\ \mathrm{r}}\right)\left(\frac{1\ \mathrm{min}}{60\ \mathrm{s}}\right) = 125.7\ \mathrm{rad/s}$$

Therefore the pullout torque of the motor is

$$\tau_{\max} = \frac{3V_{\mathrm{TH}}^2}{2\omega_{\mathrm{sync}}\left[R_{\mathrm{TH}} + \sqrt{R_{\mathrm{TH}}^2 + \left(X_{\mathrm{TH}} + X_2\right)^2}\right]}$$

$$\tau_{\max} = \frac{3(247.5\ \mathrm{V})^2}{2(125.7\ \mathrm{rad/s})\left[0.0778\,\Omega + \sqrt{(0.0778\,\Omega)^2 + (0.223\,\Omega + 0.216\,\Omega)^2}\right]}$$

$$\tau_{\max} = 1396\ \mathrm{N \cdot m}$$

7-17. Plot the following quantities for the motor in Problem 7-14 as slip varies from 0% to 10%: *(a)* τ_{ind} *(b)* P_{conv} *(c)* P_{out} *(d)* Efficiency η. At what slip does P_{out} equal the rated power of the machine?

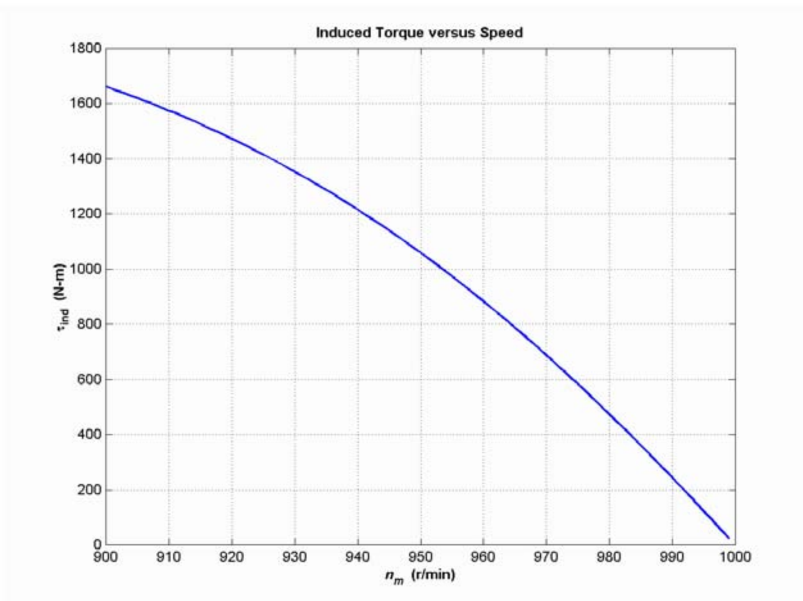
SOLUTION This problem is ideally suited to solution with a MATLAB program. An appropriate program is shown below. It follows the calculations performed for Problem 7-14, but repeats them at many values of slip, and then plots the results. Note that it plots all the specified values versus n_m, which varies from 900 to 1000 r/min, corresponding to a range of 0 to 10% slip.

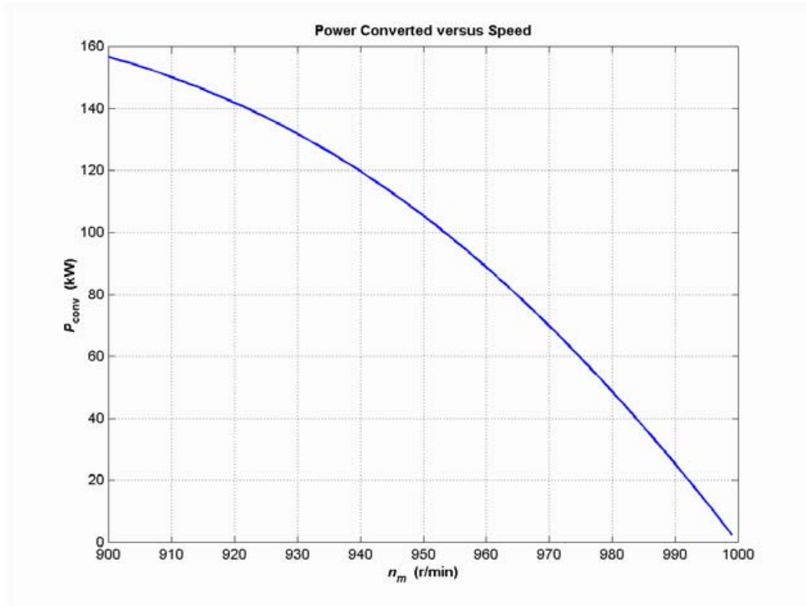
```
% M-file: prob7_17.m
% M-file create a plot of the induced torque, power
%   converted, power out, and efficiency of the induction
%   motor of Problem 7-14 as a function of slip.

% First, initialize the values needed in this program.
r1 = 0.082;                 % Stator resistance
x1 = 0.190;                 % Stator reactance
r2 = 0.070;                 % Rotor resistance
x2 = 0.180;                 % Rotor reactance
xm = 7.2;                   % Magnetization branch reactance
v_phase = 440 / sqrt(3);    % Phase voltage
n_sync = 1000;              % Synchronous speed (r/min)
w_sync = 104.7;             % Synchronous speed (rad/s)
p_mech = 1300;              % Mechanical losses (W)
p_core = 1400;              % Core losses (W)
p_misc = 150;               % Miscellaneous losses (W)

% Calculate the Thevenin voltage and impedance from Equations
```

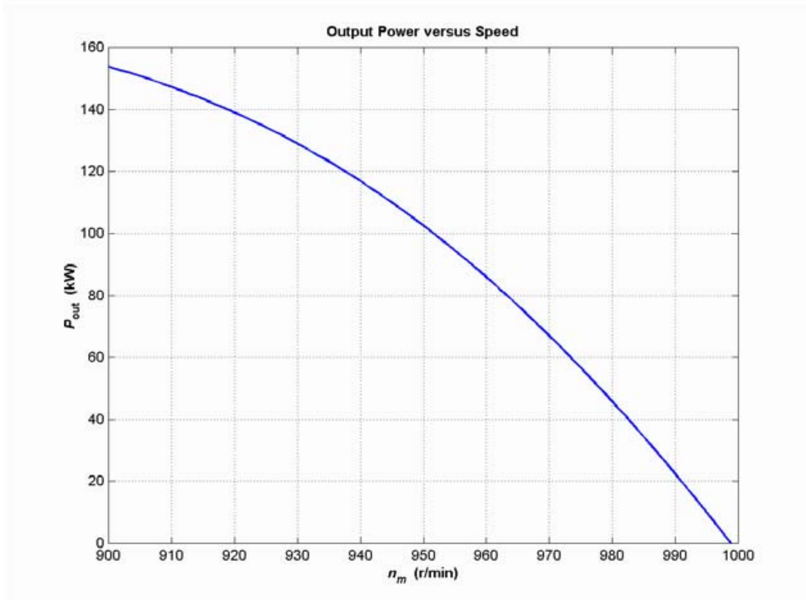
```
% 7-38 and 7-41.
v_th = v_phase * ( xm / sqrt(r1^2 + (x1 + xm)^2) );
z_th = ((j*xm) * (r1 + j*x1)) / (r1 + j*(x1 + xm));
r_th = real(z_th);
x_th = imag(z_th);

% Now calculate the torque-speed characteristic for many
% slips between 0 and 0.1.  Note that the first slip value
% is set to 0.001 instead of exactly 0 to avoid divide-
% by-zero problems.
s = (0:0.001:0.1);           % Slip
s(1) = 0.001;
nm = (1 - s) * n_sync;       % Mechanical speed
wm = nm * 2*pi/60;           % Mechanical speed

% Calculate torque, P_conv, P_out, and efficiency
% versus speed
for ii = 1:length(s)

   % Induced torque
   t_ind(ii) = (3 * v_th^2 * r2 / s(ii)) / ...
           (w_sync * ((r_th + r2/s(ii))^2 + (x_th + x2)^2) );

   % Power converted
   p_conv(ii) = t_ind(ii) * wm(ii);

   % Power output
   p_out(ii) = p_conv(ii) - p_mech - p_core - p_misc;

   % Power input
   zf = 1 / ( 1/(j*xm) + 1/(r2/s(ii)+j*x2) );
   ia = v_phase / ( r1 + j*x1 + zf );
   p_in(ii) = 3 * v_phase * abs(ia) * cos(atan(imag(ia)/real(ia)));

   % Efficiency
   eff(ii) = p_out(ii) / p_in(ii) * 100;

end

% Plot the torque-speed curve
figure(1);
plot(nm,t_ind,'b-','LineWidth',2.0);
xlabel('\bf\itn_{m}  \rm\bf(r/min)');
ylabel('\bf\tau_{ind}  \rm\bf(N-m)');
title ('\bfInduced Torque versus Speed');
grid on;

% Plot power converted versus speed
figure(2);
plot(nm,p_conv/1000,'b-','LineWidth',2.0);
xlabel('\bf\itn_{m}  \rm\bf(r/min)');
ylabel('\bf\itP\rm\bf_{conv}  (kW)');
title ('\bfPower Converted versus Speed');
grid on;

% Plot output power versus speed
```

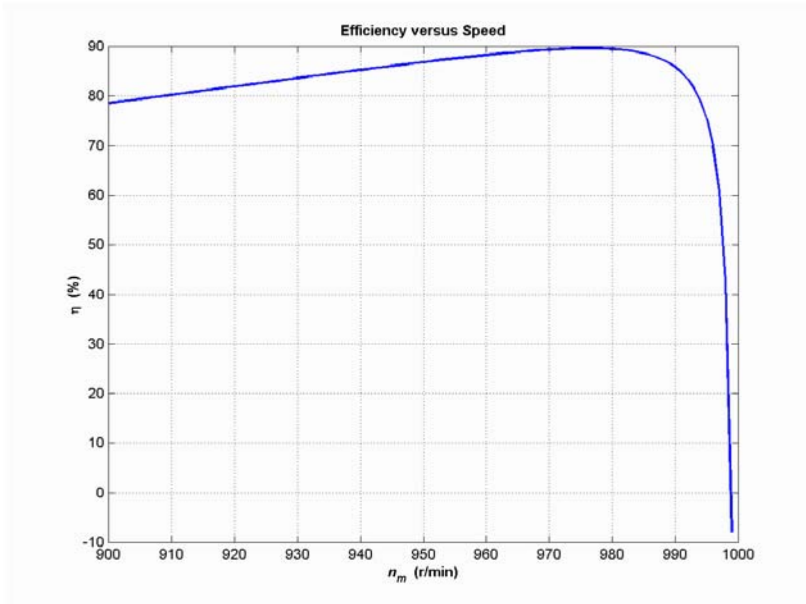
```
figure(3);
plot(nm,p_out/1000,'b-','LineWidth',2.0);
xlabel('\bf\itn_{m}  \rm\bf(r/min)');
ylabel('\bf\itP\rm\bf_{out}  (kW)');
title ('\bfOutput Power versus Speed');
axis([900 1000 0 160]);
grid on;

% Plot the efficiency
figure(4);
plot(nm,eff,'b-','LineWidth',2.0);
xlabel('\bf\itn_{m}  \rm\bf(r/min)');
ylabel('\bf\eta  (%)');
title ('\bfEfficiency versus Speed');
grid on;
```

The four plots are shown below:




Power Converted versus Speed
160
140
120
100
80
60
40
20
0
900
910
920
930
940
950
960
970
980
990
1000
P_{conv} (kW)
n_m (r/min)


Output Power versus Speed
160
140
120
100
80
60
40
20
0
900
910
920
930
940
950
960
970
980
990
1000
P_{out} (kW)
n_m (r/min)



This machine is rated at 75 kW. It produces an output power of 75 kW at 3.4% slip, or a speed of 966 r/min.

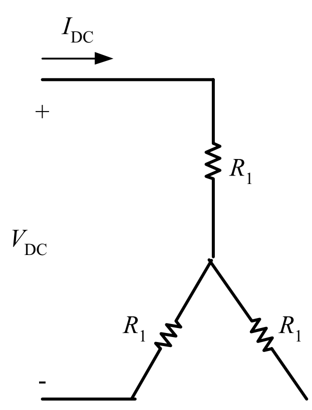
7-18. A 208-V, 60 Hz, six-pole Y-connected 25-hp design class B induction motor is tested in the laboratory, with the following results:

No load:	208 V, 22.0 A, 1200 W, 60 Hz
Locked rotor:	24.6 V, 64.5 A, 2200 W, 15 Hz
DC test:	13.5 V, 64 A

Find the equivalent circuit of this motor, and plot its torque-speed characteristic curve.

SOLUTION From the DC test,

$$2R_1 = \frac{13.5\text{ V}}{64\text{ A}} \qquad \Rightarrow \qquad R_1 = 0.105\,\Omega$$



In the no-load test, the line voltage is 208 V, so the phase voltage is 120 V. Therefore,

$$X_1 + X_M = \frac{V_\phi}{I_{A,\text{nl}}} = \frac{120\text{ V}}{22.0\text{ A}} = 5.455\,\Omega \quad @ \quad 60\text{ Hz}$$

In the locked-rotor test, the line voltage is 24.6 V, so the phase voltage is 14.2 V. From the locked-rotor test *at 15 Hz*,

$$|Z'_{\mathrm{LR}}| = |R_{\mathrm{LR}} + jX'_{\mathrm{LR}}| = \frac{V_\phi}{I_{A,\mathrm{LR}}} = \frac{14.2 \text{ V}}{64.5 \text{ A}} = 0.2202\ \Omega$$

$$\theta'_{\mathrm{LR}} = \cos^{-1}\frac{P_{\mathrm{LR}}}{S_{\mathrm{LR}}} = \cos^{-1}\left(\frac{2200 \text{ W}}{\sqrt{3}(24.6 \text{ V})(64.5 \text{ A})}\right) = 36.82°$$

Therefore,

$$R_{\mathrm{LR}} = |Z'_{\mathrm{LR}}|\cos\theta_{\mathrm{LR}} = (0.2202\ \Omega)\cos 36.82° = 0.176\ \Omega$$

$$\Rightarrow \quad R_1 + R_2 = 0.176\ \Omega$$

$$\Rightarrow \quad R_2 = 0.071\ \Omega$$

$$X'_{\mathrm{LR}} = |Z'_{\mathrm{LR}}|\sin\theta_{\mathrm{LR}} = (0.2202\ \Omega)\sin 36.82° = 0.132\ \Omega$$

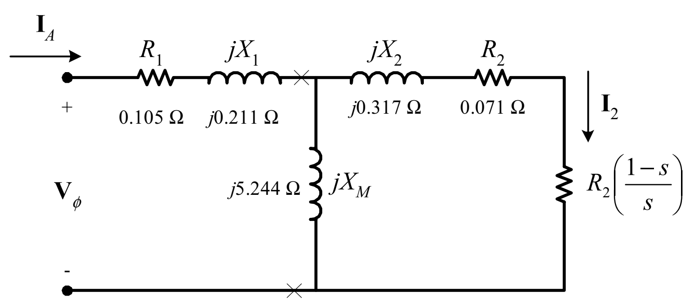
At a frequency of 60 Hz,

$$X_{\mathrm{LR}} = \left(\frac{60 \text{ Hz}}{15 \text{ Hz}}\right) X'_{\mathrm{LR}} = 0.528\ \Omega$$

For a Design Class B motor, the split is $X_1 = 0.211\ \Omega$ and $X_2 = 0.317\ \Omega$. Therefore,

$$X_M = 5.455\ \Omega - 0.211\ \Omega = 5.244\ \Omega$$

The resulting equivalent circuit is shown below:

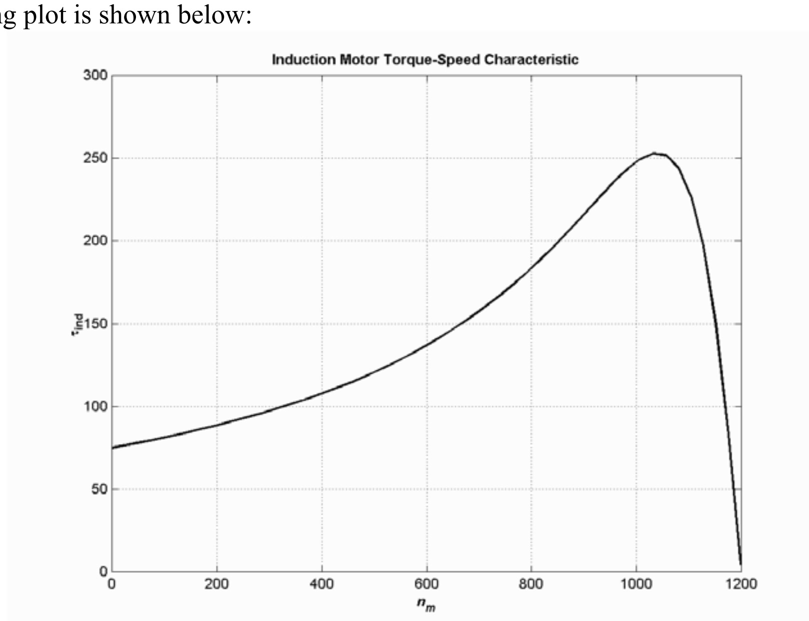


A MATLAB program to calculate the torque-speed characteristic of this motor is shown below:

```
% M-file: prob7_18.m
% M-file create a plot of the torque-speed curve of the
%   induction motor of Problem 7-18.

% First, initialize the values needed in this program.
r1 = 0.105;                 % Stator resistance
x1 = 0.211;                 % Stator reactance
r2 = 0.071;                 % Rotor resistance
x2 = 0.317;                 % Rotor reactance
xm = 5.244;                 % Magnetization branch reactance
v_phase = 208 / sqrt(3);    % Phase voltage
n_sync = 1200;              % Synchronous speed (r/min)
w_sync = 125.7;             % Synchronous speed (rad/s)

% Calculate the Thevenin voltage and impedance from Equations
% 7-38 and 7-41.
```

```
v_th = v_phase * ( xm / sqrt(r1^2 + (x1 + xm)^2) );
z_th = ((j*xm) * (r1 + j*x1)) / (r1 + j*(x1 + xm));
r_th = real(z_th);
x_th = imag(z_th);

% Now calculate the torque-speed characteristic for many
% slips between 0 and 1.  Note that the first slip value
% is set to 0.001 instead of exactly 0 to avoid divide-
% by-zero problems.
s = (0:1:50) / 50;          % Slip
s(1) = 0.001;
nm = (1 - s) * n_sync;      % Mechanical speed

% Calculate torque versus speed
for ii = 1:51
   t_ind(ii) = (3 * v_th^2 * r2 / s(ii)) / ...
            (w_sync * ((r_th + r2/s(ii))^2 + (x_th + x2)^2) );
end

% Plot the torque-speed curve
figure(1);
plot(nm,t_ind,'b-','LineWidth',2.0);
xlabel('\bf\itn_{m}');
ylabel('\bf\tau_{ind}');
title ('\bfInduction Motor Torque-Speed Characteristic');
grid on;
```

The resulting plot is shown below:



7-19. A 208-V four-pole 10-hp 60-Hz Y-connected three-phase induction motor develops its full-load induced torque at 3.8 percent slip when operating at 60 Hz and 208 V. The per-phase circuit model impedances of the motor are

$R_1 = 0.33\ \Omega$ $\qquad X_M = 16\ \Omega$

$X_1 = 0.42\ \Omega$ $\qquad X_2 = 0.42\ \Omega$

Mechanical, core, and stray losses may be neglected in this problem.

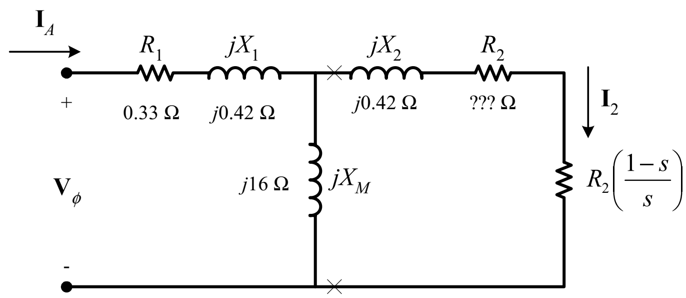
(a) Find the value of the rotor resistance R_2.

(b) Find $\tau_{\max}$, $s_{\max}$, and the rotor speed at maximum torque for this motor.

(c) Find the starting torque of this motor.

(d) What code letter factor should be assigned to this motor?

SOLUTION The equivalent circuit for this motor is



The Thevenin equivalent of the input circuit is:

$$Z_{\text{TH}} = \frac{jX_M\left(R_1 + jX_1\right)}{R_1 + j\left(X_1 + X_M\right)} = \frac{\left(j16\ \Omega\right)\left(0.33\ \Omega\ + j0.42\ \Omega\right)}{0.33\ \Omega\ + j\left(0.42\ \Omega + 16\ \Omega\right)} = 0.313 + j0.416\ \Omega = 0.520\angle 53°\ \Omega$$

$$\mathbf{V}_{\text{TH}} = \frac{jX_M}{R_1 + j\left(X_1 + X_M\right)}\mathbf{V}_\phi = \frac{\left(j16\,\Omega\right)}{0.33\,\Omega + j\left(0.42\,\Omega + 16\,\Omega\right)}\left(120\angle 0°\text{ V}\right) = 116.9\angle 1.2°\text{ V}$$

(a) If losses are neglected, the induced torque in a motor is equal to its load torque. At full load, the output power of this motor is 10 hp and its slip is 3.8%, so the induced torque is

$$n_m = \left(1 - 0.038\right)\left(1800\text{ r/min}\right) = 1732\text{ r/min}$$

$$\tau_{\text{ind}} = \tau_{\text{load}} = \frac{\left(10\text{ hp}\right)\left(746\text{ W/hp}\right)}{\left(1732\text{ r/min}\right)\left(\dfrac{2\pi\text{ rad}}{1\text{ r}}\right)\left(\dfrac{60\text{ s}}{1\text{ min}}\right)} = 41.1\text{ N}\cdot\text{m}$$

The induced torque is given by the equation

$$\tau_{\text{ind}} = \frac{3V_{\text{TH}}^2 R_2/s}{\omega_{\text{sync}}\left[\left(R_{\text{TH}} + R_2/s\right)^2 + \left(X_{\text{TH}} + X_2\right)^2\right]}$$

Substituting known values and solving for R_2/s yields

$$41.1\text{ N}\cdot\text{m} = \frac{3\left(116.9\text{ V}\right)^2 R_2/s}{\left(188.5\text{ rad/s}\right)\left[\left(0.313 + R_2/s\right)^2 + \left(0.416 + 0.42\right)^2\right]}$$

$$7{,}747 = \frac{40{,}997\ \ R_2/s}{\left[\left(0.313 + R_2/s\right)^2 + 0.699\right]}$$

$$\left[\left(0.313 + R_2/s\right)^2 + 0.699\right] = 5.292\ \ R_2/s$$

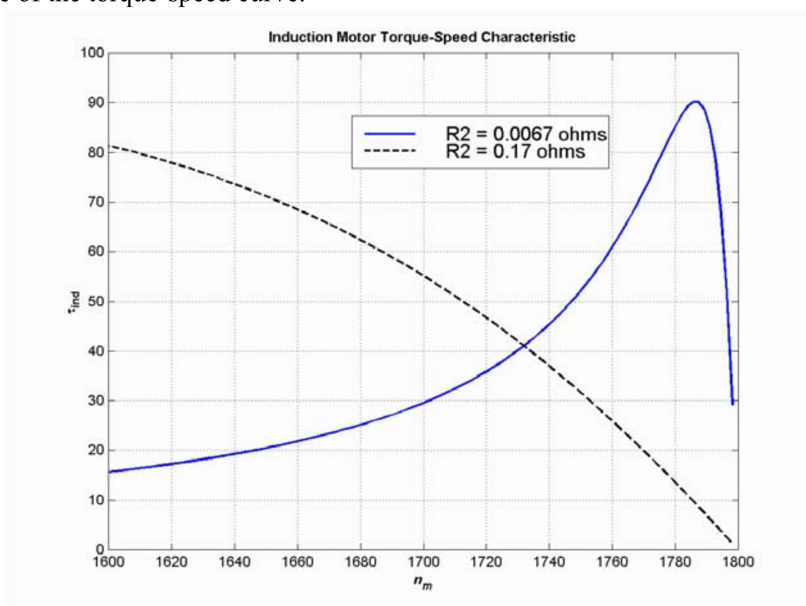
$$\left[0.098+0.626R_2/s+\left(R_2/s\right)^2+0.699\right]=5.292\ \ R_2/s$$

$$\left(\frac{R_2}{s}\right)^2-4.666\left(\frac{R_2}{s}\right)+0.797=0$$

$$\left(\frac{R_2}{s}\right)=0.178,\ \ 4.488$$

$$R_2=0.0067\,\Omega,\ 0.17\,\Omega$$

These two solutions represent two situations in which the torque-speed curve would go through this specific torque-speed point. The two curves are plotted below. As you can see, only the 0.17 Ω solution is realistic, since the 0.0067 Ω solution passes through this torque-speed point at an unstable location on the back side of the torque-speed curve.



(b) The slip at pullout torque can be found by calculating the Thevenin equivalent of the input circuit from the rotor back to the power supply, and then using that with the rotor circuit model. The Thevenin equivalent of the input circuit was calculate in part *(a)*. The slip at pullout torque is

$$s_{\max}=\frac{R_2}{\sqrt{{R_{\mathrm{TH}}}^2+\left(X_{\mathrm{TH}}+X_2\right)^2}}$$

$$s_{\max}=\frac{0.17\ \Omega}{\sqrt{\left(0.313\ \Omega\right)^2+\left(0.416\ \Omega\ +0.420\ \Omega\right)^2}}=0.190$$

The rotor speed a maximum torque is

$$n_{\mathrm{pullout}}=(1-s)\ n_{\mathrm{sync}}=\left(1-0.190\right)\left(1800\ \mathrm{r/min}\right)=1457\ \mathrm{r/min}$$

and the pullout torque of the motor is

$$\tau_{\max}=\frac{3V_{\mathrm{TH}}^2}{2\omega_{\mathrm{sync}}\left[R_{\mathrm{TH}}+\sqrt{R_{\mathrm{TH}}^2+\left(X_{\mathrm{TH}}+X_2\right)^2}\right]}$$

$$\tau_{\max} = \frac{3(116.9\ \text{V})^2}{2(188.5\ \text{rad/s})\left[0.313\ \Omega + \sqrt{(0.313\ \Omega)^2 + (0.416\ \Omega + 0.420\ \Omega)^2}\right]}$$

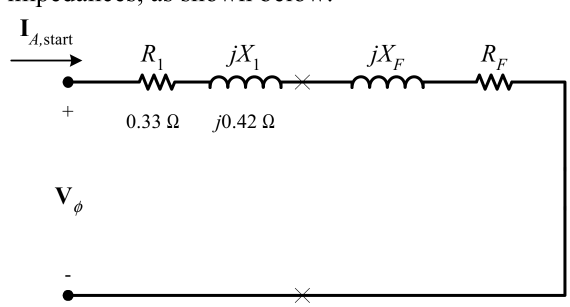
$$\tau_{\max} = 90.2\ \text{N}\cdot\text{m}$$

(c) The starting torque of this motor is the torque at slip $s = 1$. It is

$$\tau_{\text{ind}} = \frac{3V_{\text{TH}}^2 R_2 / s}{\omega_{\text{sync}}\left[\left(R_{\text{TH}} + R_2/s\right)^2 + \left(X_{\text{TH}} + X_2\right)^2\right]}$$

$$\tau_{\text{ind}} = \frac{3(116.9\ \text{V})^2(0.17\ \Omega)}{(188.5\ \text{rad/s})\left[(0.313 + 0.17\ \Omega)^2 + (0.416 + 0.420)^2\right]} = 38.3\ \text{N}\cdot\text{m}$$

(d) To determine the starting code letter, we must find the locked-rotor kVA per horsepower, which is equivalent to finding the starting kVA per horsepower. The easiest way to find the line current (or armature current) at starting is to get the equivalent impedance Z_F of the rotor circuit in parallel with jX_M at starting conditions, and then calculate the starting current as the phase voltage divided by the sum of the series impedances, as shown below.



The equivalent impedance of the rotor circuit in parallel with jX_M at starting conditions ($s = 1.0$) is:

$$Z_{F,\text{start}} = \frac{1}{\dfrac{1}{jX_M} + \dfrac{1}{Z_2}} = \frac{1}{\dfrac{1}{j16\ \Omega} + \dfrac{1}{0.17 + j0.42}} = 0.161 + j0.411 = 0.442\angle 68.6^\circ\ \Omega$$

The phase voltage is $208/\sqrt{3} = 120$ V, so line current $\mathbf{I}_{L,\text{start}}$ is

$$\mathbf{I}_{L,\text{start}} = \mathbf{I}_A = \frac{\mathbf{V}_\phi}{R_1 + jX_1 + R_F + jX_F} = \frac{120\angle 0^\circ\ \text{V}}{0.33\ \Omega + j0.42\ \Omega + 0.161\ \Omega + j0.411\ \Omega}$$

$$\mathbf{I}_{L,\text{start}} = \mathbf{I}_A = 124\angle -59.4^\circ\ \text{A}$$

Therefore, the locked-rotor kVA of this motor is

$$S = \sqrt{3}\, V_T\, I_{L,\text{rated}} = \sqrt{3}(208\ \text{V})(124\ \text{A}) = 44.7\ \text{kVA}$$

and the kVA per horsepower is

$$\text{kVA/hp} = \frac{44.7\ \text{kVA}}{10\ \text{hp}} = 4.47\ \text{kVA/hp}$$

This motor would have **starting code letter D**, since letter D covers the range 4.00-4.50.

7-20. Answer the following questions about the motor in Problem 7-19.

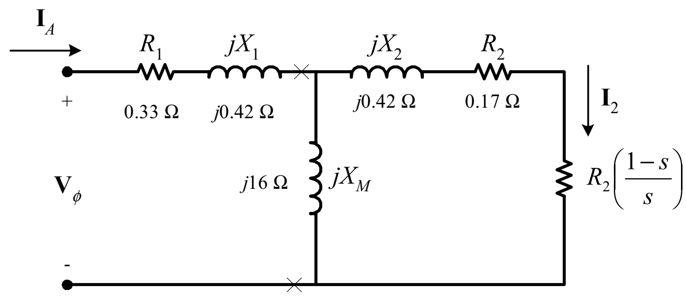
(a) If this motor is started from a 208-V infinite bus, how much current will flow in the motor at starting?

(b) If transmission line with an impedance of 0.50 + *j*0.35 Ω per phase is used to connect the induction motor to the infinite bus, what will the starting current of the motor be? What will the motor's terminal voltage be on starting?

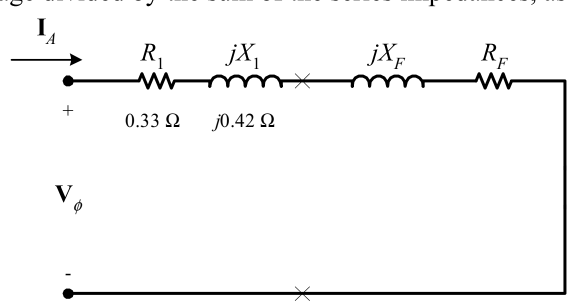
(c) If an ideal 1.2:1 step-down autotransformer is connected between the transmission line and the motor, what will the current be in the transmission line during starting? What will the voltage be at the motor end of the transmission line during starting?

SOLUTION

(a) The equivalent circuit of this induction motor is shown below:



The easiest way to find the line current (or armature current) at starting is to get the equivalent impedance Z_F of the rotor circuit in parallel with jX_M at starting conditions, and then calculate the starting current as the phase voltage divided by the sum of the series impedances, as shown below.



The equivalent impedance of the rotor circuit in parallel with jX_M at starting conditions (*s* = 1.0) is:

$$Z_F = \frac{1}{\dfrac{1}{jX_M} + \dfrac{1}{Z_2}} = \frac{1}{\dfrac{1}{j16\,\Omega} + \dfrac{1}{0.17 + j0.42}} = 0.161 + j0.411 = 0.442\angle 68.6^\circ\,\Omega$$

The phase voltage is 208/$\sqrt{3}$ = 120 V, so line current $\mathbf{I}_L$ is

$$\mathbf{I}_L = \mathbf{I}_A = \frac{\mathbf{V}_\phi}{R_1 + jX_1 + R_F + jX_F} = \frac{120\angle 0^\circ \text{ V}}{0.33\,\Omega + j0.42\,\Omega + 0.161\,\Omega + j0.411\,\Omega}$$

$$\mathbf{I}_L = \mathbf{I}_A = 124\angle -59.4^\circ \text{ A}$$

(b) If a transmission line with an impedance of 0.50 + *j*0.35 Ω per phase is used to connect the induction motor to the infinite bus, its impedance will be in series with the motor's impedances, and the starting current will be

$$\mathbf{I}_L = \mathbf{I}_A = \frac{\mathbf{V}_{\phi,\text{bus}}}{R_{\text{line}} + jX_{\text{line}} + R_1 + jX_1 + R_F + jX_F}$$

$$\mathbf{I}_L = \mathbf{I}_A = \frac{120\angle 0° \text{ V}}{0.50\ \Omega\ + j0.35\ \Omega + 0.33\ \Omega\ + j0.42\ \Omega +\ 0.161\ \Omega + j0.411\ \Omega}$$

$$\mathbf{I}_L = \mathbf{I}_A = 77.8\angle -50.0° \text{ A}$$

The voltage at the terminals of the motor will be

$$\mathbf{V}_\phi = \mathbf{I}_A\left(R_1 + jX_1 + R_F + jX_F\right)$$

$$\mathbf{V}_\phi = \left(77.8\angle -50.0° \text{ A}\right)\left(0.33\ \Omega\ + j0.42\ \Omega +\ 0.161\ \Omega + j0.411\ \Omega\right)$$

$$\mathbf{V}_\phi = 75.1\angle 9.4° \text{ V}$$

Therefore, the terminal voltage will be $\sqrt{3}\ \left(75.1 \text{ V}\right) = 130 \text{ V}$. Note that the terminal voltage sagged by 37.5% during motor starting, which would be unacceptable.

(c) If an ideal 1.2:1 step-down autotransformer is connected between the transmission line and the motor, the motor's impedances will be referred across the transformer by the square of the turns ratio a = 1.2. The referred impedances are

$$R_1' = a^2 R_1 = 1.44\left(0.33\ \Omega\right) = 0.475\ \Omega$$

$$X_1' = a^2 X_1 = 1.44\left(0.42\ \Omega\right) = 0.605\ \Omega$$

$$R_F' = a^2 R_F = 1.44\left(0.161\ \Omega\right) = 0.232\ \Omega$$

$$X_F' = a^2 X_F = 1.44\left(0.411\ \Omega\right) = 0.592\ \Omega$$

Therefore, the starting current referred to the primary side of the transformer will be

$$\mathbf{I}_L' = \mathbf{I}_A' = \frac{\mathbf{V}_{\phi,\text{bus}}}{R_{\text{line}} + jX_{\text{line}} + R_1' + jX_1' + R_F' + jX_F'}$$

$$\mathbf{I}_L' = \mathbf{I}_A' = \frac{120\angle 0° \text{ V}}{0.50\ \Omega + j0.35\ \Omega + 0.475\ \Omega + j0.605\ \Omega + 0.232\ \Omega + j0.592\ \Omega}$$

$$\mathbf{I}_L' = \mathbf{I}_A' = 61.2\angle -52° \text{ A}$$

The voltage at the motor end of the transmission line would be the same as the referred voltage at the terminals of the motor

$$\mathbf{V}_\phi' = \mathbf{I}_A'\left(R_1' + jX_1' + R_F' + jX_F'\right)$$

$$\mathbf{V}_\phi = \left(61.2\angle -52° \text{ A}\right)\left(0.475\ \Omega + j0.605\ \Omega + 0.232\ \Omega + j0.592\ \Omega\right)$$

$$\mathbf{V}_\phi = 85.0\angle 7.4° \text{ V}$$

Therefore, the line voltage at the motor end of the transmission line will be $\sqrt{3}\ \left(85 \text{ V}\right) = 147.3 \text{ V}$. Note that this voltage sagged by 29.2% during motor starting, which is less than the 37.5% sag with case of across-the-line starting. Since the sag is still large, it might be possible to use a bigger autotransformer turns ratio on the starter.

7-21. In this chapter, we learned that a step-down autotransformer could be used to reduce the starting current drawn by an induction motor. While this technique works, an autotransformer is relatively expensive. A much less expensive way to reduce the starting current is to use a device called *Y-Δ starter*. If an induction motor is normally Δ-connected, it is possible to reduce its phase voltage V_ϕ (and hence its starting current) by simply re-connecting the stator windings in Y during starting, and then restoring the connections to Δ when the motor comes up to speed. Answer the following questions about this type of starter.

(a) How would the phase voltage at starting compare with the phase voltage under normal running conditions?

(b) How would the starting current of the Y-connected motor compare to the starting current if the motor remained in a Δ-connection during starting?

SOLUTION

(a) The phase voltage at starting would be $1 / \sqrt{3}$ = 57.7% of the phase voltage under normal running conditions.

(b) Since the phase voltage decreases to $1 / \sqrt{3}$ = 57.7% of the normal voltage, the starting phase current will also decrease to 57.7% of the normal starting current. However, since the line current for the original delta connection was $\sqrt{3}$ times the phase current, while the line current for the Y starter connection is equal to its phase current, *the line current is reduced by a factor of 3* in a Y-Δ starter.

For the Δ-connection: $I_{L,\Delta} = \sqrt{3}\, I_{\phi,\Delta}$

For the Y-connection: $I_{L,\mathrm{Y}} = I_{\phi,\mathrm{Y}}$

But $I_{\phi,\Delta} = \sqrt{3} I_{\phi,\mathrm{Y}}$, so $\boxed{I_{L,\Delta} = 3 I_{L,\mathrm{Y}}}$

7-22. A 460-V 50-hp six-pole Δ-connected 60-Hz three-phase induction motor has a full-load slip of 4 percent, an efficiency of 91 percent, and a power factor of 0.87 lagging. At start-up, the motor develops 1.75 times the full-load torque but draws 7 times the rated current at the rated voltage. This motor is to be started with an autotransformer reduced voltage starter.

(a) What should the output voltage of the starter circuit be to reduce the starting torque until it equals the rated torque of the motor?

(b) What will the motor starting current and the current drawn from the supply be at this voltage?

SOLUTION

(a) The starting torque of an induction motor is proportional to the square of V_{TH} ,

$$\frac{\tau_{\mathrm{start2}}}{\tau_{\mathrm{start1}}} = \left(\frac{V_{\mathrm{TH2}}}{V_{\mathrm{TH1}}}\right)^2 = \left(\frac{V_{T2}}{V_{T1}}\right)^2$$

$$\frac{\tau_{\mathrm{start2}}}{\tau_{\mathrm{start1}}} = \left(\frac{V_{\mathrm{TH2}}}{V_{\mathrm{TH1}}}\right)^2 = \left(\frac{V_{T2}}{V_{T1}}\right)^2$$

If a torque of 1.75 τ_{rated} is produced by a voltage of 460 V, then a torque of 1.00 τ_{rated} would be produced by a voltage of

$$\frac{1.00\,\tau_{\mathrm{rated}}}{1.75\,\tau_{\mathrm{rated}}} = \left(\frac{V_{T2}}{460\ \mathrm{V}}\right)^2$$

$$V_{T2} = \sqrt{\frac{(460\ \mathrm{V})^2}{1.75}} = 348\ \mathrm{V}$$

(b) The motor starting current is directly proportional to the starting voltage, so

$$I_{L2} = \left(\frac{348\ \mathrm{V}}{460\ \mathrm{V}}\right) I_{L1} = (0.756) I_{L1} = (0.756)\left(7 I_{\mathrm{rated}}\right) = 5.296\, I_{\mathrm{rated}}$$

The input power to this motor is

$$P_{\text{IN}} = \frac{P_{\text{OUT}}}{\eta} = \frac{(50\text{ hp})(746\text{ W/hp})}{0.91} = 40.99\text{ kW}$$

The rated current is equal to

$$I_{\text{rated}} = \frac{P_{\text{IN}}}{\sqrt{3}\ V_T\ \text{PF}} = \frac{(40.99\text{ kW})}{\sqrt{3}(460\text{ V})(0.87)} = 59.1\text{ A}$$

Therefore, the motor starting current is

$$I_{L2} = 5.843\, I_{\text{rated}} = (5.296)(59.1\text{ A}) = 313\text{ A}$$

The turns ratio of the autotransformer that produces this starting voltage is

$$\frac{N_{SE} + N_C}{N_C} = \frac{460\text{ V}}{348\text{ V}} = 1.32$$

so the current drawn from the supply will be

$$I_{\text{line}} = \frac{I_{\text{start}}}{1.32} = \frac{313\text{ A}}{1.32} = 237\text{ A}$$

7-23. A wound-rotor induction motor is operating at rated voltage and frequency with its slip rings shorted and with a load of about 25 percent of the rated value for the machine. If the rotor resistance of this machine is doubled by inserting external resistors into the rotor circuit, explain what happens to the following:

(a) Slip s

(b) Motor speed n_m

(c) The induced voltage in the rotor

(d) The rotor current

(e) τ_{ind}

(f) P_{out}

(g) P_{RCL}

(h) Overall efficiency η

SOLUTION

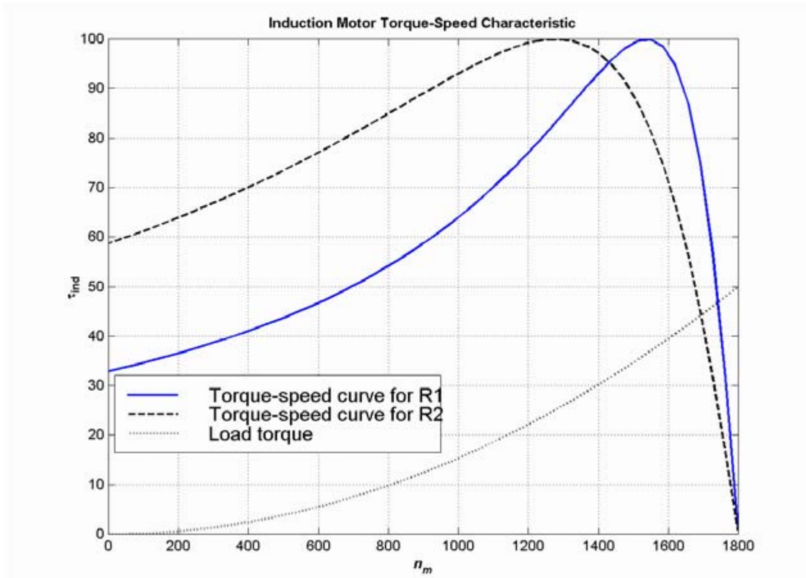
(a) The slip s will increase.

(b) The motor speed n_m will decrease.

(c) The induced voltage in the rotor will increase.

(d) The rotor current will increase.

(e) The induced torque will adjust to supply the load's torque requirements at the new speed. This will depend on the shape of the load's torque-speed characteristic. For most loads, the induced torque will decrease.



(f) The output power will generally decrease: $P_{\text{OUT}} = \tau_{\text{ind}} \downarrow \omega_m \downarrow$

(g) The rotor copper losses (including the external resistor) will increase.

(h) The overall efficiency η will decrease.

7-24. Answer the following questions about a 460-V Δ-connected two-pole 100-hp 60-Hz starting code letter F induction motor:

(a) What is the maximum current starting current that this machine's controller must be designed to handle?

(b) If the controller is designed to switch the stator windings from a Δ connection to a Y connection during starting, what is the maximum starting current that the controller must be designed to handle?

(c) If a 1.25:1 step-down autotransformer starter is used during starting, what is the maximum starting current that will be drawn from the line?

SOLUTION

(a) The maximum starting kVA of this motor is

$$S_{\text{start}} = (100 \text{ hp})(5.60) = 560 \text{ kVA}$$

Therefore,

$$I_{\text{start}} = \frac{S}{\sqrt{3}\,V_T} = \frac{560 \text{ kVA}}{\sqrt{3}\,(460 \text{ V})} = 703 \text{ A}$$

(b) The line voltage will still be 460 V when the motor is switched to the Y-connection, but now the phase voltage will be 460 / $\sqrt{3}$ = 265.6 V.

<u>Before (in Δ):</u>

$$I_{\phi,\Delta} = \frac{460 \text{ V}}{(R_{\text{TH}} + R_2) + j(X_{\text{TH}} + X_2)}$$

$$I_{L,\Delta} = \sqrt{3} I_{\phi,\Delta} = \frac{797\text{ V}}{(R_{\text{TH}} + R_2) + j(X_{\text{TH}} + X_2)}$$

After (in Y):

$$I_{L,\text{Y}} = I_{\phi,\text{Y}} = \frac{265.6\text{ V}}{(R_{\text{TH}} + R_2) + j(X_{\text{TH}} + X_2)}$$

Therefore the line current will decrease by a factor of 3! The starting current with a Δ-Y starter is

$$I_{\text{start}} = \frac{703\text{ A}}{3} = 234\text{ A}$$

(c) A 1.25:1 step-down autotransformer reduces the phase voltage on the motor by a factor 0.8. This reduces the phase current and line current in the motor (and on the secondary side of the transformer) by a factor of 0.8. However, the current on the primary of the autotransformer will be reduced by another factor of 0.8, so the total starting current drawn from the line will be 64% of its original value. Therefore, the maximum starting current drawn from the line will be

$$I_{\text{start}} = (0.64)(703\text{ A}) = 450\text{ A}$$

7-25. When it is necessary to stop an induction motor very rapidly, many induction motor controllers reverse the direction of rotation of the magnetic fields by switching any two stator leads. When the direction of rotation of the magnetic fields is reversed, the motor develops an induced torque opposite to the current direction of rotation, so it quickly stops and tries to start turning in the opposite direction. If power is removed from the stator circuit at the moment when the rotor speed goes through zero, then the motor has been stopped very rapidly. This technique for rapidly stopping an induction motor is called *plugging*. The motor of Problem 7-19 is running at rated conditions and is to be stopped by plugging.

(a) What is the slip s before plugging?

(b) What is the frequency of the rotor before plugging?

(c) What is the induced torque τ_{ind} before plugging?

(d) What is the slip s immediately after switching the stator leads?

(e) What is the frequency of the rotor immediately after switching the stator leads?

(f) What is the induced torque τ_{ind} immediately after switching the stator leads?

SOLUTION

(a) The slip before plugging is 0.038 (see Problem 7-19).

(b) The frequency of the rotor before plugging is $f_r = sf_e = (0.038)(60\text{ Hz}) = 2.28\text{ Hz}$

(c) The induced torque before plugging is 41.1 N·m in the direction of motion (see Problem 7-19).

(d) After switching stator leads, the synchronous speed becomes –1800 r/min, while the mechanical speed initially remains 1732 r/min. Therefore, the slip becomes

$$s = \frac{n_{\text{sync}} - n_m}{n_{\text{sync}}} = \frac{-1800 - 1732}{-1800} = 1.962$$

(e) The frequency of the rotor after plugging is $f_r = sf_e = (1.962)(60\text{ Hz}) = 117.72\text{ Hz}$

(f) The induced torque immediately after switching the stator leads is

$$\tau_{\text{ind}} = \frac{3V_{\text{TH}}^2 R_2 / s}{\omega_{\text{sync}}\left[\left(R_{\text{TH}} + R_2 / s\right)^2 + \left(X_{\text{TH}} + X_2\right)^2\right]}$$

$$\tau_{\text{ind}} = \frac{3\left(116.9 \text{ V}\right)^2 \left(0.17 \ \Omega/1.962\right)}{\left(188.5 \text{ rad/s}\right)\left[\left(0.313 + 0.17 \ \Omega/1.962\right)^2 + \left(0.416 + 0.420\right)^2\right]}$$

$$\tau_{\text{ind}} = \frac{3\left(116.9 \text{ V}\right)^2 \left(0.0866\right)}{\left(188.5 \text{ rad/s}\right)\left[\left(0.313 + 0.0866\right)^2 + \left(0.416 + 0.420\right)^2\right]}$$

$$\tau_{\text{ind}} = 21.9 \text{ N} \cdot \text{m, opposite the direction of motion}$$