EE 221
Circuits II

Chapter 9
Sinusoids and Phasors

Sinusoids and Phasors

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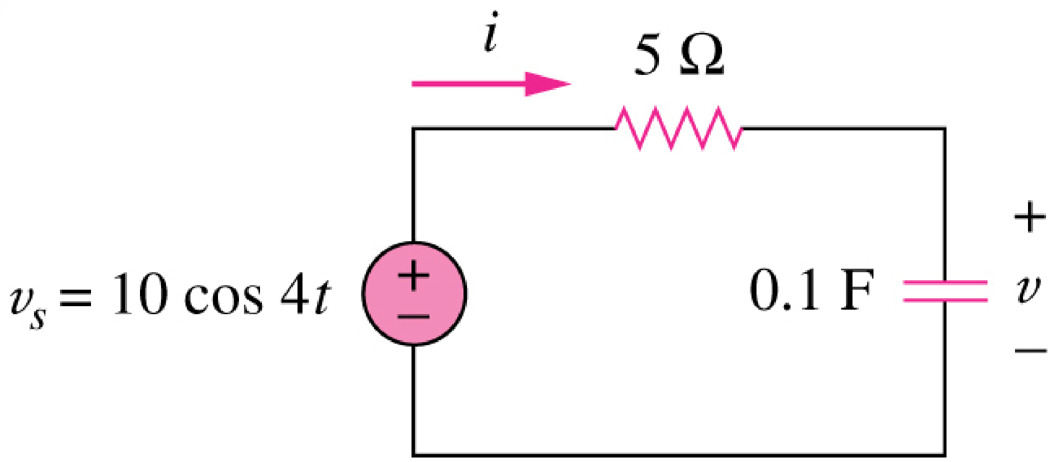
9.1 Motivation

How to determine *v(t)* and *i(t) at steady-state*?



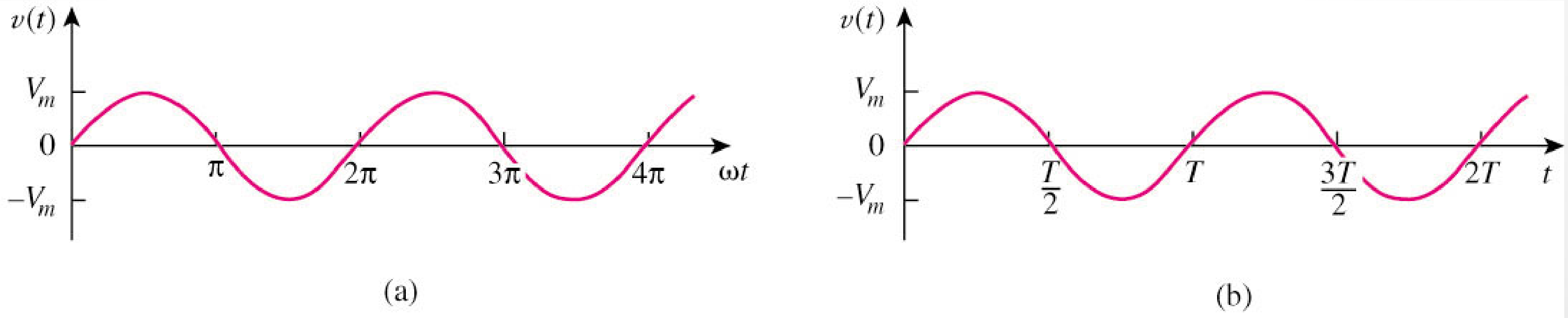
First order circuit (Chap. 7), apply KVL to determine the equation, write the general expression of the solution, interested only in the steady-state or forced response.

Is there a quicker way?

9.2 Sinusoids

- A sinusoid is a signal that has the form of the sine or cosine function.
- A general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi)$$

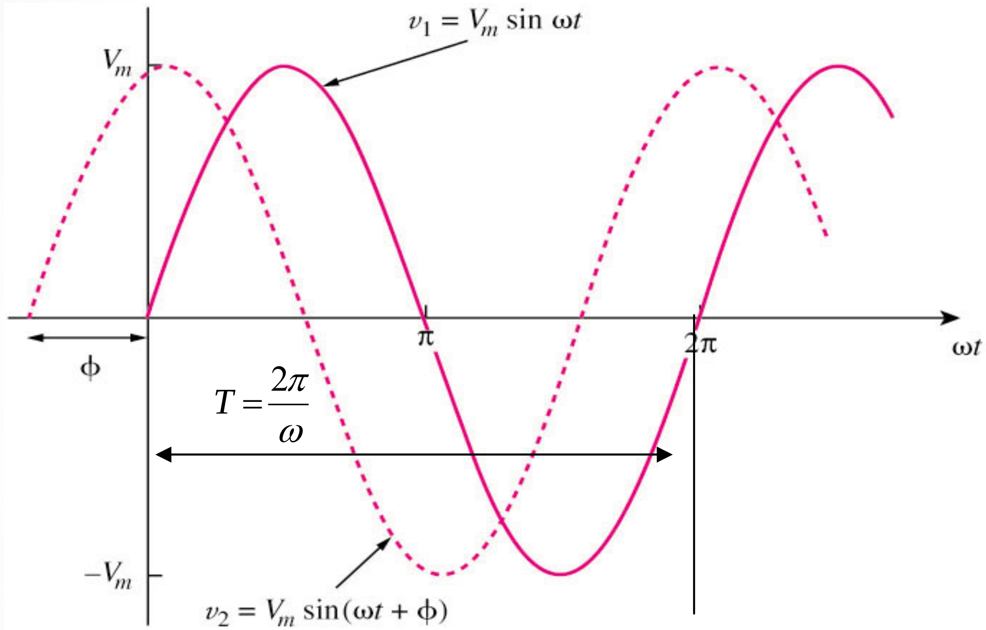


where

Vm = the **amplitude** of the sinusoid
ω = the angular frequency in radians/s
Φ = the phase angle.

9.2 Sinusoids

A <u>periodic function</u> is one that satisfies $v(t) = v(t + nT)$, for all t and for all integers n.



$$f = \frac{1}{T} Hz \quad \omega = 2\pi f$$

- Only two sinusoidal values with the <u>same frequency</u> can be compared by their amplitude and phase difference.
- If phase difference is zero, they are in phase; if phase difference is not zero, they are out of phase.

9.2 Sinusoids

Example 1

Given a sinusoid, $5\sin(4\pi t - 60^o)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Solution:

amplitude = 5,
phase = –60^o,
angular frequency = 4π rad/s,
period = 0.5 s,
Linear frequency = 2 Hz.

9.2 Sinusoids

Example 2

Find the phase angle between $i_1 = -4\sin(377t + 25^o)$ and $i_2 = 5\cos(377t - 40^o)$, does i_1 lead or lag i_2?

Solution:

Since $\sin(\omega t+90^o) = \cos \omega t$

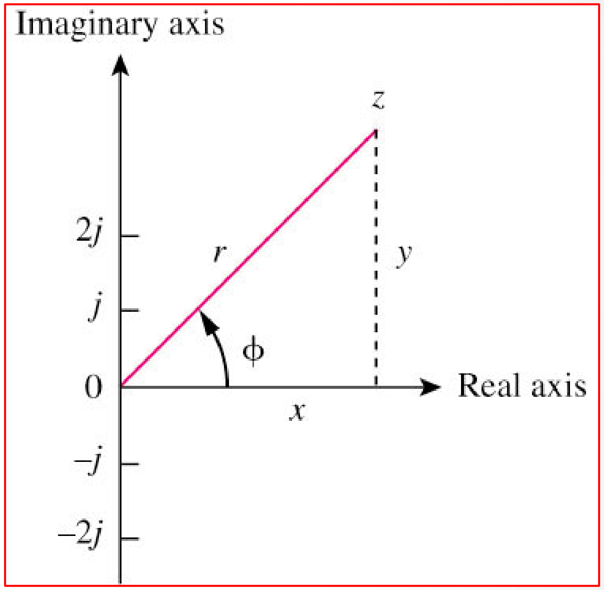
$$i_2 = 5\sin(377t - 40^o + 90^o) = 5\sin(377t + 50^o)$$

$$i_1 = -4\sin(377t + 25^o) = 4\sin(377t + 180^o + 25^o) = 4\sin(377t + 205^o)$$

therefore, i_1 leads i_2 by 155^o.

9.3 Phasors

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.
- It can be represented in one of the following three forms:



a. Rectangular $z = x + jy = r(\cos\phi + j\sin\phi)$

b. Polar $z = r\angle\phi$

c. Exponential $z = re^{j\phi}$

where $r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\frac{y}{x}$

9.3 Phasors

Mathematic operation of complex number:

1. Addition $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
3. Multiplication $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$
4. Division $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \angle \phi_1 - \phi_2$
5. Reciprocal $\dfrac{1}{z} = \dfrac{1}{r} \angle -\phi$
6. Square root $\sqrt{z} = \sqrt{r} \angle \phi/2$
7. Complex conjugat $z^* = x - jy = r \angle -\phi = re^{-j\phi}$
8. Euler's identity $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

9.3 Phasors

Example 3

- Evaluate the following complex numbers:

a. $[(5+j2)(-1+j4)-5\angle 60^{o}]$

b. $\dfrac{10+j5+3\angle 40^{o}}{-3+j4}+10\angle 30^{o}$

Solution:

a. –15.5 + j13.67

b. 8.293 + j2.2

9.3 Phasors

- Transform a sinusoid to and from the time domain to the phasor domain:

$$v(t) = V_m \cos(\omega t + \phi) \longleftrightarrow V = V_m \angle \phi$$

(time domain) (phasor domain)

- Amplitude and phase difference are two principal concerns in the study of voltage and current sinusoids.

- Phasor will be defined from the cosine function in all our proceeding study. If a voltage or current expression is in the form of a sine, it will be changed to a cosine by subtracting 90 deg from the phase.

9.3 Phasors

Example 4

Transform the following sinusoids to phasors:

$i = 6\cos(50t - 40^o)$ A

$v = -4\sin(30t + 50^o)$ V

Solution:

a. $I = 6\angle -40^\circ$ A

b. Since $-\sin(A) = \cos(A+90^o)$;

$v(t) = 4\cos(30t + 50^o + 90^o) = 4\cos(30t + 140^o)$ V

Transform to phasor → $V = 4\angle 140^\circ$ V

9.3 Phasors

Example 5:

Transform the phasors to corresponding sinusoids:

a. $\mathbf{V} = -10\angle 30^\circ \text{ V}$

b. $\mathbf{I} = j(5 - j12) \text{ A}$

Solution:

a) $v(t) = 10\cos(\omega t + 210^o)$ V

b) Since $\mathbf{I} = 12 + j5 = \sqrt{12^2 + 5^2} \angle \tan^{-1}(\frac{5}{12}) = 13\angle 22.62^\circ$

$i(t) = 13\cos(\omega t + 22.62^o)\ A$

9.3 Phasors

The differences between v(t) and V:

- v(t) is instantaneous or time-domain representation **V** is the frequency or phasor-domain representation.
- v(t) is time dependent, **V** is not.
- v(t) is always real with no complex term, **V** is generally complex.

Note: Phasor analysis applies only when frequency is constant; when it is applied to two or more sinusoid signals only if they have the same frequency.

9.3 Phasors (derivatives and integrals)

$$v(t) = V_m \cos(\omega t + \phi) = \Re\{V_m e^{j(\omega t + \phi)}\}$$
$$= \Re\{V_m e^{j\omega t} e^{j\phi}\} = \Re\{e^{j\omega t} V_m \angle \phi\} \Rightarrow V_m \angle \phi$$

$$dv(t)/dt = d[V_m \cos(\omega t + \phi)]/dt = d[\Re\{V_m e^{j(\omega t + \phi)}\}]/dt$$
$$= d[\Re\{V_m e^{j\omega t} e^{j\phi}\}]/dt = j\omega \Re\{e^{j\omega t} V_m \angle \phi\} \Rightarrow j\omega V_m \angle \phi$$

$$\int v(t)dt = \int V_m \cos(\omega t + \phi)dt = \int \Re\{V_m e^{j(\omega t + \phi)}\}dt$$
$$= \int \Re\{V_m e^{j\omega t} e^{j\phi}\}dt = \Re\{e^{j\omega t} V_m \angle \phi\}/j\omega \Rightarrow \frac{1}{j\omega} V_m \angle \phi$$

9.3 Phasors

Example 6

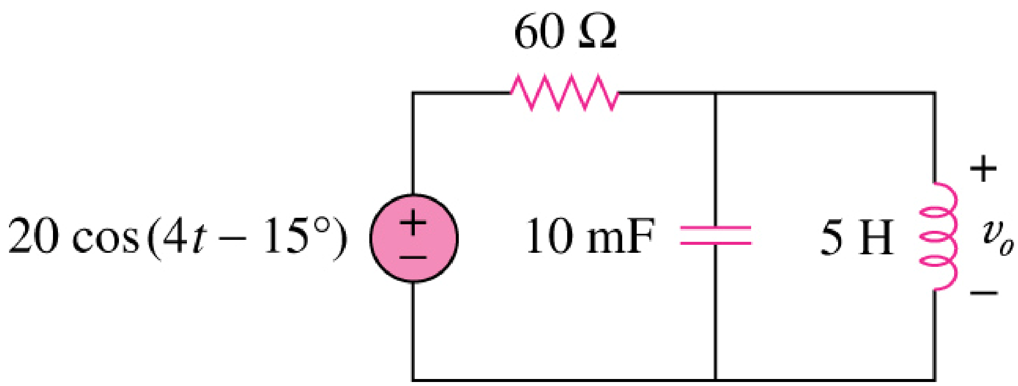
Use phasor approach, determine the current *i(t)* in a circuit described by the integro-differential equation.

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75°)$$

Answer: $i(t) = 4.642\cos(2t + 143.2^{o})\ A$

9.3 Phasors

- We can derive the differential equations for the following circuit, then transform it in the phasor domain, solve for $\mathbf{V_o}$, then transform back in the time domain to find $v_o(t)$. However, the derivation may sometimes be very tedious



$$\frac{d^2 v_o}{dt^2} + \frac{5}{3}\frac{dv_0}{dt} + 20v_0 = -\frac{400}{3}\sin(4t - 15^o)$$

Is there any quicker and more systematic methods to solve the voltage across the inductor?

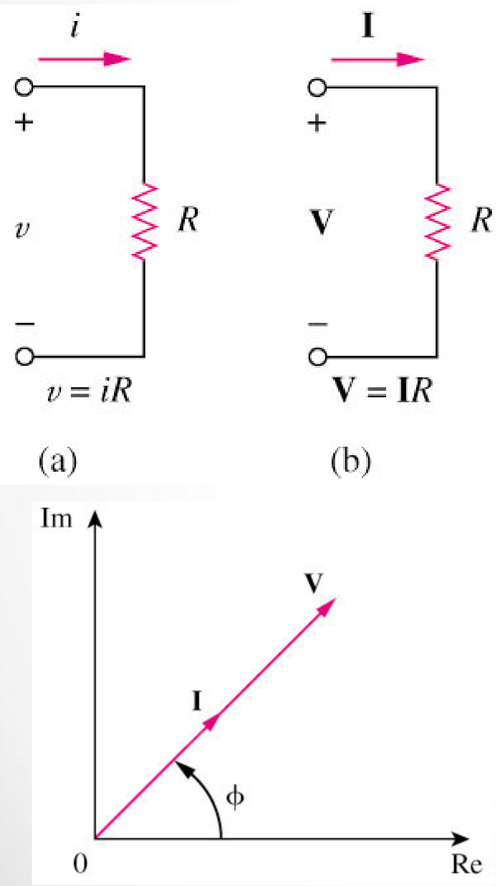
9.3 Phasors

The answer is YES!

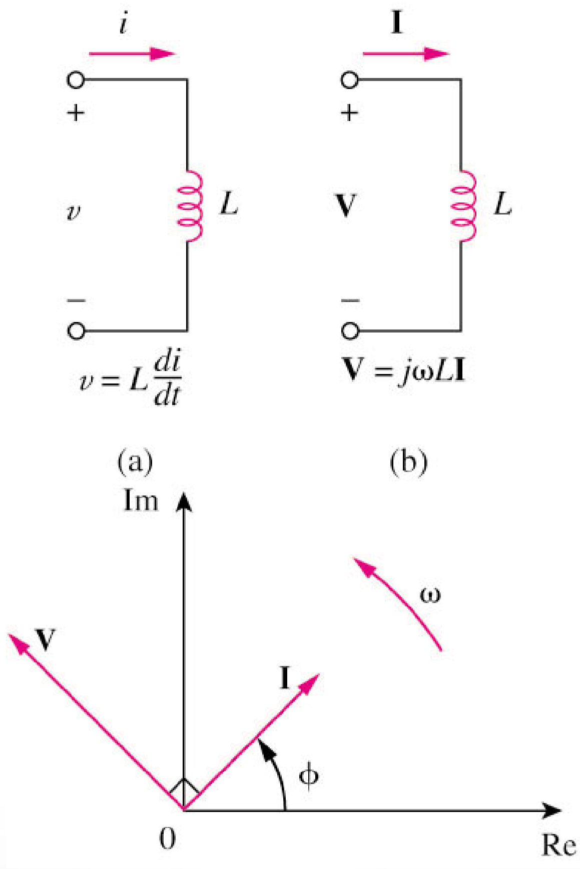
Instead of first deriving the differential equation and then transforming it into phasor to solve for V_{o}, we can transform all the RLC components into phasor the phasor domain first, then apply the KCL laws and other theorems to set up a phasor equation involving V_o directly.

9.4 Phasor Relationships for Circuit Elements

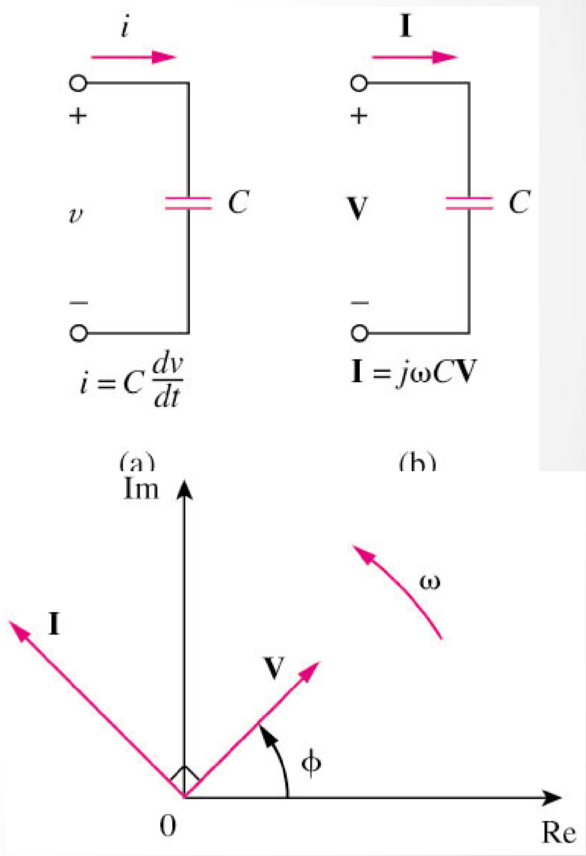
Resistor:



Inductor:



Capacitor:



9.4 Phasor Relationships for Circuit Elements

Summary of voltage-current relationship		
Element	Time domain	Frequency domain
R	$v = Ri$	$V = RI$
L	$v = L\dfrac{di}{dt}$	$V = j\omega LI$
C	$i = C\dfrac{dv}{dt}$	$V = \dfrac{I}{j\omega C}$

9.4 Phasor Relationships for Circuit Elements

Example 7

If voltage $v(t) = 6\cos(100t - 30^o)$ is applied to a 50 μF capacitor, calculate the current, $i(t)$, through the capacitor.

Answer: $i(t) = 30 \cos(100t + 60^o)$ mA

9.5 Impedance and Admittance

- The impedance **Z** of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in Ω.

$$Z = \frac{V}{I} = R \pm jX$$

where R = *Re* (Z) is the resistance and X = Im (Z) is the reactance. Positive (+) is for inductance and negative (-) is for capacitance.

- The admittance Y is the reciprocal of impedance, measured in (S).

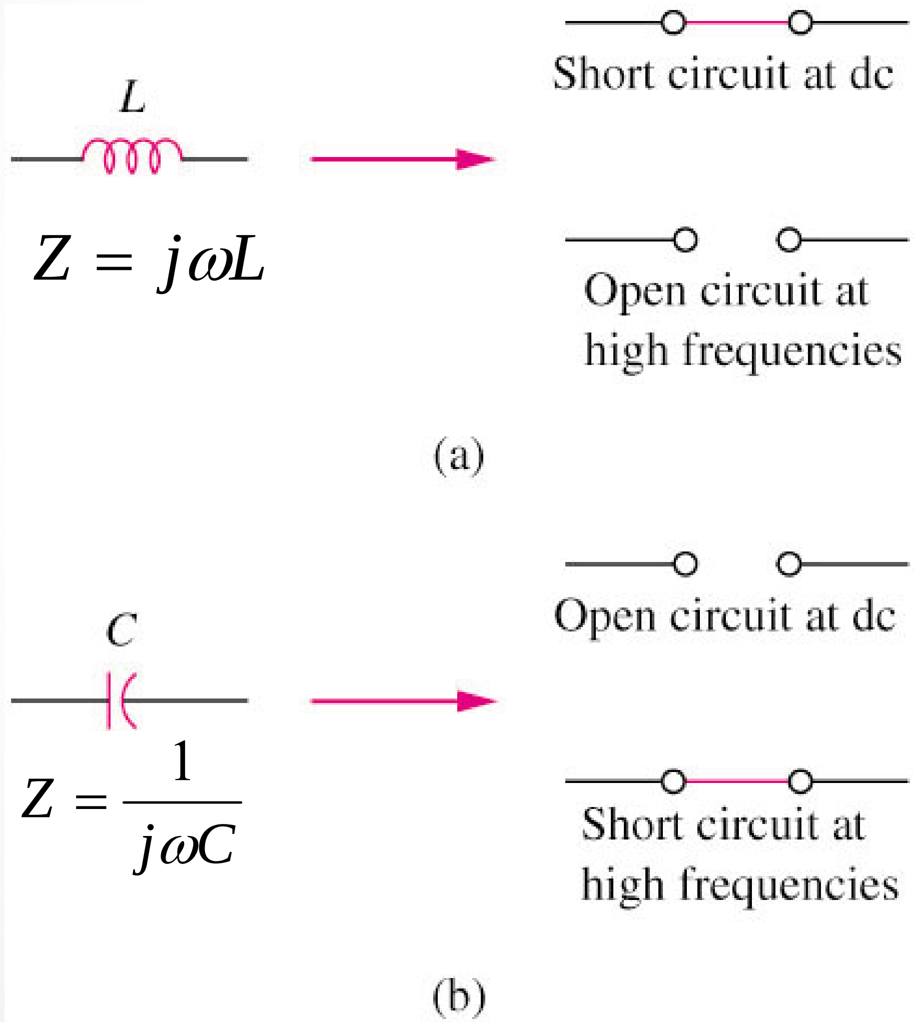
$$Y = \frac{1}{Z} = \frac{I}{V} = G \pm jB$$

where G = *Re* (Y) is the conductance and B = Im (Y) is the suceptance. Positive (+) is for capacitance and negative (-) is for inductance.

9.5 Impedance and Admittance

Impedances and admittances of passive elements		
Element	**Impedance**	**Admittance**
R	$Z = R$	$Y = \dfrac{1}{R}$
L	$Z = j\omega L$	$Y = \dfrac{1}{j\omega L}$
C	$Z = \dfrac{1}{j\omega C}$	$Y = j\omega C$

9.5 Impedance and Admittance



$$\omega = 0; Z = 0$$

$$\omega \rightarrow \infty; Z \rightarrow \infty$$

$$\omega = 0; Z \rightarrow \infty$$

$$\omega \rightarrow \infty; Z = 0$$

9.5 Impedance and Admittance

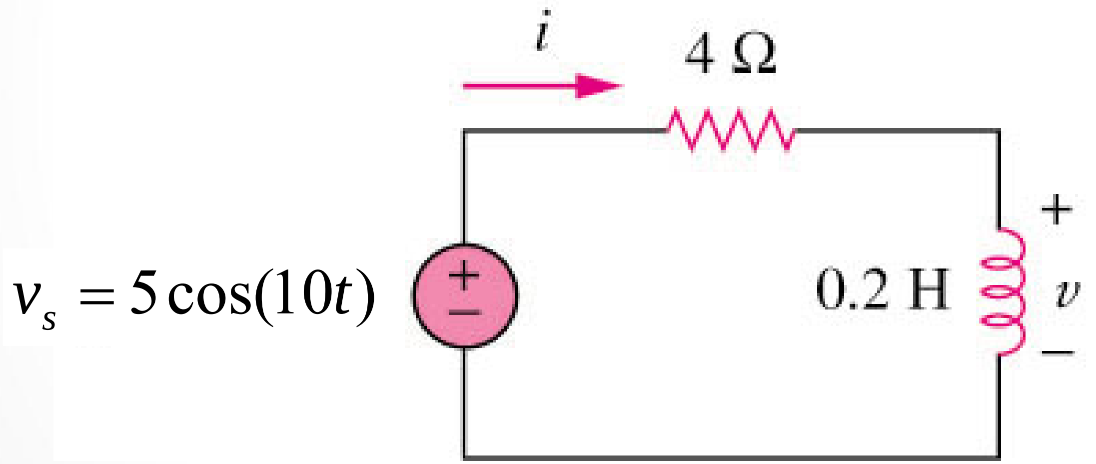
After we know how to convert RLC components from the time domain to thw phasor domain, we can transform a time domain circuit into a phasor/frequency domain circuit.

Hence, we can apply the KCL laws and other theorems to directly set up phasor equations involving our target variable(s) for solving.

9.5 Impedance and Admittance

Example 8

Refer to Figure below, determine *v(t)* and *i(t).*



Answers: $i(t) = 1.118\cos(10t - 26.56^o)$ A;

$v(t) = 2.236\cos(10t + 63.43^o)$ V

9.6 Kirchhoff's Laws in the Frequency Domain

- Both KVL and KCL are hold in the phasor domain or more commonly called frequency domain.

- Moreover, the variables to be handled are phasors, which are complex numbers.

- All the mathematical operations involved are now in complex form.

9.7 Impedance Combinations

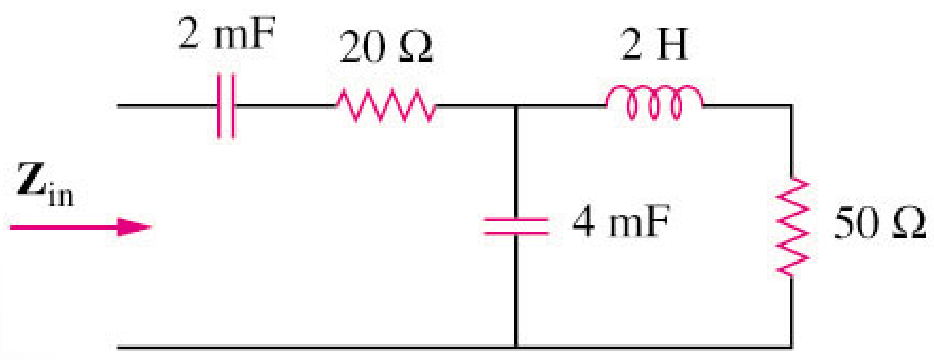
- All principles used for DC circuit analysis also apply to AC circuit.

- For example:
  a. voltage division
  b. current division
  c. Series – parallel combinations
  d. Y-Δ transformation

9.7 Impedance Combinations

Example 9

Determine the input impedance of the circuit in figure below at ω =10 rad/s.



Answer: Z_{in} = **32.38 – j73.76 Ω**