1. $A = \sin(2 - 3i)$

$$= \frac{e^{i(2-3i)} - e^{-i(2-3i)}}{2i}$$

$$= \frac{1}{2i}\left(e^{3+2i} - e^{-3-2i}\right)$$

$$= \frac{-i}{2}\left(e^3 \cos 2 + ie^3 \sin 2 - e^{-3} \cos 2 + ie^{-3} \sin 2\right)$$

$$A= \boxed{cosh3sin2-\text{i}sinh3cosh2}$$

2. $\operatorname{arcsinh} z = 1 + i$

$$z = \frac{e^{1+i} - e^{-1-i}}{2}$$

$$= \frac{e(\cos 1 + i \sin 1) - e^{-1}(\cos 1 - i \sin 1)}{2}, \qquad e^z = e^x(\cos y + i \sin y)$$

$$z = \frac{e \cos 1 - \frac{\cos 1}{e}}{2} + i\frac{e \sin 1 + \frac{\sin 1}{e}}{2}$$

$$z = \boxed{\cos 1 \sinh 1 + i \sin 1 \cosh 1}$$

3. $f(z) = z^3 \cos z - \sin z + ze^{-z^2}$

- $$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \qquad z^3 \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+3}}{(2n)!}$$
- $$-\sin z = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n-1)!}$$
- $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad ze^{-z^2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!}$$

fix powers: go to 2^{n+1}

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n+1}}{(2n-2)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(n)!}$$

fix indices: set to 1

$$f(z) = -z + z \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{(2n-2)!} + \frac{1}{(2n+1)!} - \frac{1}{n!} \right) z^{2n+1}$$

4. For the function $V(x,y) = \sin x \cosh y$, (a) prove that V is harmonic. (b)Find U(x,y) that is conjugate harmonic to V, and (c) express $f(z) = u + iv$ in terms of only z.

(a)

$$V(x,y) = \sin x \cosh y \Rightarrow \frac{\partial v}{\partial x} = \cos x \cosh y, \qquad \frac{\partial^2 v}{\partial x^2} = -\sin x \cosh y,$$

$$\frac{\partial v}{\partial y} = \sin x \sinh y, \qquad \frac{\partial^2 v}{\partial y^2} = \sin x \cosh y,$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\sin x \cosh y + \sin x \cosh y = 0.$$

Hence V is harmonic.

(b)

By Cauchy-Riemann: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

$$U(x,y) = -\int \cos x \cosh y dy = -\cos x \int \cosh y dy$$

$$U(x,y) = -\cos x \sinh y + \phi(x).$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \sin x \sinh y + \phi'(x)$$

$$\sin x \sinh y = \sin x \sinh y + \phi'(x)$$

$$\phi'(x) = 0$$

$$\phi(x) = C$$

Hence:

$$U(x,y) = -\cos x \sinh y + C$$

(c)

$W(z) = U + iV$

$$
\begin{aligned}
&= -\cos x \sinh y + C + i \sin x \cosh y \\
&= -\cos x\left(\frac{e^y - e^{-y}}{2}\right) + i\sin x\left(\frac{e^y + e^{-y}}{2}\right) + C \\
&= -\frac{1}{2}e^y \cos x + \frac{1}{2}e^{-y}\cos x + \frac{i}{2}e^y \sin x + \frac{i}{2}e^{-y}\sin x + C \\
&= \frac{e^y}{2}\left(-\cos x + i\sin x\right) + \frac{e^{-y}}{2}\left(\cos x + i\sin x\right) + C \\
&= \frac{e^y}{2}(-e^{-ix}) + \frac{e^{-y}}{2}(e^{ix}) \\
&= \frac{-e^{y-ix}}{2} + \frac{e^{-y+ix}}{2} \\
&= \frac{-1}{2}e^{-iz} + \frac{1}{2}e^{iz} + C \\
&= \frac{-\cos z}{2} + \frac{i\sin z}{2} + \frac{\cos z}{2} + \frac{i\sin z}{2} + C \\
&= i\sin z + C
\end{aligned}
$$

$$\boxed{W(z) = i\sin z + C}$$

5. Find first three non-zero terms of Maclaurin Series $f(z) = \tan z$.

$$\tan z = \frac{\sin z}{\cos z} = \frac{z - \frac{z^3}{6} + \frac{z^5}{120} - \dots}{1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 z^4 + \dots$$

Since tan is an odd function,

$$\frac{z - \frac{z^3}{6} + \frac{z^5}{120} - \dots}{1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots} = C_1 z + C_3 z^3 + C_5 z^5 + C_7 z^7 \dots$$

$$z - \frac{z^3}{6} + \frac{z^5}{120} - \dots = \left(C_1 z + C_3 z^3 + C_5 z^5 + \dots\right)\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots\right)$$

equate like terms:

$$C_1 = 1, \qquad C_3 = \frac{1}{3}, \qquad C_5 = \frac{2}{15}$$

$$\boxed{\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \dots}$$

6. $I = \sin^3 z$

$$= \qquad \frac{\sin z - \sin z \cos 2z}{2}.$$

$$\frac{\sin z}{2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!},$$

$$\sin z \cos 2z = \frac{\sin(-z)}{2} + \frac{\sin 3z}{2} = \frac{-\sin z + \sin 3z}{2},$$

$$-\frac{\sin z \cos 2z}{2} = \frac{\sin z - \sin 3z}{4}$$

$$\boxed{I = \frac{3}{4}\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^{2n-1} z^{2n-1}}{(2n-1)!}}$$

7. Find the radius of convergence.

$$\sum_{n=7}^{\infty} \frac{(n!)^2 2^{3n+3}}{(2n+5)!!}(2z-5+4i)^n = \sum_{n=7}^{\infty} \frac{(n!)^2 2^{4n+3}}{(2n+5)!!}(z-5/2+2i)^n,$$

$$z_0 = \frac{5}{2} - 2i$$

$$\frac{1}{R} = \lim_{n\to+\infty}\left|\frac{(n+1)!^2 2^{4n+7}}{(2n+7)!!} \cdot \frac{(2n+5)!!}{(n!)^2 2^{4n+3}}\right| = 16 \lim_{n\to+\infty}\left|\frac{(n+1)^2}{2n+7}\right| = +\infty$$

$$\boxed{R = +0}$$

8. Find the radius and domain of convergence.

$$\sum_{n=7}^{\infty} \frac{i^n n! 2^{3n+3}}{(2n+5)!!}(2z-5+4i)^n = \sum_{n=7}^{\infty} \frac{i^n n! 2^{4n+3}}{(2n+5)!!}(z-5/2+2i)^n,$$

$$z_0 = \frac{5}{2} - 2i$$

$$\frac{1}{R} = \lim_{n\to+\infty}\left|\frac{i^{n+1}(n+1)! 2^{4n+7}}{(2n+7)!!} \cdot \frac{(2n+5)!!}{i^n n! 2^{4n+3}}\right| = 16 \lim_{n\to+\infty}\left|\frac{i(n+1)}{(2n+7)}\right| = 8$$

$$\boxed{R = \frac{1}{8}}$$

$$\boxed{D : \left|z - \frac{5}{2} + 2i\right| < \frac{1}{8}}$$