

EWD386

The Solution to a Cyclic Relaxation Problem

The problem solved in this note arose in connection with the (just initiated) study of self-stabilizing systems.

Consider a circle and N points numbered from 0 through $N - 1$, placed in an *arbitrary* order around the circumference. For "the adjustment of point $nr.i$ " we consider the shortest clockwise path along the circumference from its predecessor —i.e. point $nr.(i - 1) \bmod N$ — to its successor —i.e. point $nr.(i + 1) \bmod N$ —; the new position of point $nr.i$ will be halfway (i.e. the middle) of that path. In formula (taking the circumference of the circle as unit) with $pred = x[(i - 1) \bmod N]$ and $succ = x[(i + 1) \bmod N]$

$$x[i] := \text{if } pred \leq succ \text{ then } (pred + succ)/2 \\ \text{else } ((pred + succ + 1)/2) \bmod 1 \text{ fi}$$

If we start doing adjustments, will the system converge to a stable state? This is not necessarily the case if we do the adjustments simultaneously, i.e. determine all the new positions in terms of all the old ones, as is shown by the following examples.

$N = 3$				$N = 4$			
t	0	1	2	t	0	1	2
$x[0]$	0	$1/2$	0	$x[0]$	0	$3/4$	0
$x[1]$	$2/3$	$1/6$	$2/3$	$x[1]$	0	$1/4$	0
$x[2]$	$1/3$	$5/6$	$1/3$	$x[2]$	$1/2$	$1/4$	$1/2$
				$x[3]$	$1/2$	$3/4$	$1/2$

For both odd and even N we have an example that will oscillate with a period 2. If, however, we do the adjustments one at a time in a fair random order (i.e. without permanent neglect of certain points), then the system is bound to converge.

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Consider, instead of the N points, the N clockwise paths leading from each point to its successor. After a point adjustment the two paths meeting at that point will both be less than $1/2$ and no future adjustment can ever undo that! After adjustments all around the circle each path will be less than $1/2$, and from that moment onwards each triangle " $i - 1, i, i + 1$ " is a clockwise one. The total clockwise path from 0 to 1, from 1 to 2, ..., from $N - 1$ to 0 will go around the circle a fixed number of times, m say ($0 \leq m \leq N/2$). No adjustment can anymore change the value of m , from now onwards we could even do simultaneous adjustments. The final state will satisfy for all i

$$(x[(i + 1) \bmod N] - x[i]) \bmod 1 = m/N$$

The system converges linearly (imagine successive points connected by spiral springs or rubber bands of equal length).

The above was written under the assumption that along the circumference we had the continuum at our disposal, i.e. that the fractions x , satisfying $0 \leq x < 1$, could be represented in arbitrary precision. Suppose now that we have to represent the fractions x as integer multiples of $1/p$ (where we may assume the integer p to satisfy $p \gg N$).

The extent to which the system converges seems to depend on how we round off when necessary, i.e. when the clockwise path from predecessor to successor turns out to be an odd multiple of $1/p$. If we impose the rule, that rounding will always take place in the same cyclic direction (say "anti-clockwise"), then the following will happen.

With m defined as in the continuous case, define q and r by

$$m * p = q * N + r \quad \text{with } 0 \leq r < N$$

Of the paths $p[i]$ leading from $x[i]$ to its successor, $(N - r)$ —the "short" paths— will be of length q/p and r paths —the "long" paths— will be of length $(q + 1)/p$. If $p[i - 1]$ is "long" and $p[i]$ is "short", adjustment of point $nr.i$ —with anti-clockwise rounding— will have the effect that the predicates "short" and "long" have interchanged position. The short ones will be travelling anti-clockwise through the cycle, simultaneously the long ones will travel clockwise through the cycle.

The two types of paths travelling in opposite direction through the same cycle makes it quite clear that if $m * p$ is an integer multiple of N , the system will converge to a completely stable situation.

Nuenen, 8th August 1973

DR. EDSGER W. DIJKSTRA

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