**Principle of Virtual Work in Dynamics**

We know from D’Alembert’s Principle that,
\[ F^i - m^i a^i = 0 \]
\[ M^i - J^i \ddot{\theta}^i = 0 \]

\textit{Newton’s Equations/ Euler’s Equation}

If \( F_i \) is selected to be at the center of mass of the body, the above equations can be multiplies by a virtual displacement and a virtual rotation without affecting them,
\[
(F^i - m^i a^i)^T \delta R^i = 0 \\
(M^i - J^i \ddot{\theta}^i) \delta \theta^i = 0
\]

where, \( R_i \) is the global position vector of the center of mass of body \( i \).

Adding these two equations,
\[
(F^i - m^i a^i)^T \delta R^i + (M^i - J^i \ddot{\theta}^i) \delta \theta^i = 0
\]

The equation can be rearranged,
\[
(F^i - m^i a^i)^T \delta R^i + (M^i - J^i \ddot{\theta}^i) \delta \theta^i = 0
\]

Or,
\[
\delta W^i - \delta W'_i = 0
\]

where, \( \delta W^i \) is the virtual work of the external and reaction forces of body \( i \)

\( \delta W'_i \) is the virtual work of the inertial forces of body \( i \)

Similarly, \( \delta W^i \) can be rewritten as
\[
\delta W^i = \delta W^i_c + \delta W^i_e
\]

where, \( \delta W^i_c \) is the virtual work of the constraint forces of body \( i \)

\( \delta W^i_e \) is the virtual work of the external forces of body \( i \)

Therefore,
\[
\delta W^i_c + \delta W^i_e - \delta W'_i = 0
\]
Connectivity Conditions
If a mechanical system is composed of \( n_b \) bodies, the equation of the previous section can be rewritten as,

\[
\sum_{i=1}^{n_b} \left( \delta W_c^i + \delta W_e^i - \delta W_l^i \right) = 0
\]

Since the joint constraint forces that act on any two adjacent bodies are equal in magnitude and opposite in direction,

\[
\sum_{i=1}^{n_b} \delta W_c^i = 0
\]

Therefore,

\[
\sum_{i=1}^{n_b} \delta W_e^i - \sum_{i=1}^{n_b} \delta W_l^i = 0
\]

or,

\[
Q_e^T \delta q - Q_l^T \delta q = 0
\]

This leads to the following dynamic equation for an interconnected body

\[
Q_e^T = Q_l^T
\]
Example:
Derive the dynamic equations for this two-link manipulator using the principle of virtual work.

Note: Gravity is active in the negative Y direction

Step 1: Develop Virtual Work Equations
The virtual work equation of the external forces is:
\[ \delta W_e = M^2 \delta \theta^2 - m^2 g \delta R^2_y - m^3 g \delta R^3_y + F^3 T \delta r^3 P \]
The virtual work of the inertial forces can be expressed as,
\[ \delta W_i = \left[ m^2 \ddot{R}_x \ m^2 \ddot{R}_y \right] \delta R^2 + f^2 \dot{\theta}^2 \delta \theta^2 + \left[ m^3 \ddot{R}_x \ m^3 \ddot{R}_y \right] \delta R^3 + f^3 \dot{\theta}^3 \delta \theta^3 \]

Step 2: Develop Virtual Displacement Equations
Based on virtual displacement analysis,
\[ \delta q = B_i \delta q_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \theta^2 \\ \delta \theta^3 \end{bmatrix} \]
and,
\[ \delta r^3 P = \begin{bmatrix} -l^2 \sin \theta^2 \\ l^2 \cos \theta^2 \\ -l^3 \sin \theta^3 \\ l^3 \cos \theta^3 \end{bmatrix} \begin{bmatrix} \delta \theta^2 \\ \delta \theta^3 \end{bmatrix} \]
Step 3: Develop Acceleration Equations
The acceleration terms can be obtained from the kinematic analysis of Chapter 3 using this equation:

\[ C_q \ddot{q} = - (C_q \dot{q})_q \dot{q} - 2C_{qt} \dot{q} - C_{tt} = Q_d \]

Since,

\[ C_{qt} = C_{tt} = 0 \]
\[ \ddot{q} = C_q^{-1} \left( - (C_q \dot{q})_q \dot{q} \right) \]

Step 4: Substitute Virtual Displacement Equations into the Virtual Work Equation

\[ \delta W_e = M^2 \delta \theta^2 - m^2 g \left( \frac{l^2}{2} \cos \theta^2 \delta \theta^2 \right) - m^3 g \left( l^2 \cos \theta^2 \delta \theta^2 + \frac{l^3}{2} \cos \theta^3 \delta \theta^3 \right) \]
\[ + \left[ F^3 \cos \phi \quad F^3 \sin \phi \right] \left[ \begin{array}{c} -l^2 \sin \theta^2 \\ l^2 \cos \theta^2 \end{array} \right] \left[ \begin{array}{c} \delta \theta^2 \\ \delta \theta^3 \end{array} \right] \]

\[ \delta W_i = \left[ m^2 \ddot{R}_x \quad m^2 \ddot{R}_y \right] \left[ \begin{array}{c} -\frac{l^2}{2} \sin \theta^2 \\ \frac{l^2}{2} \cos \theta^2 \end{array} \right] \delta \theta^2 + F^2 \ddot{\theta} \delta \theta^2 \]
\[ + \left[ m^3 \ddot{R}_x \quad m^3 \ddot{R}_y \right] \left[ \begin{array}{c} -l^2 \sin \theta^2 \\ l^2 \cos \theta^2 \end{array} \right] \left[ \begin{array}{c} \frac{l^3}{2} \sin \theta^3 \\ \frac{l^3}{2} \cos \theta^3 \end{array} \right] \delta \theta^3 + J^3 \ddot{\theta}^3 \delta \theta^3 \]

Rearranging,

\[ \delta W_e = \left( M^2 - m^2 g \left( \frac{l^2}{2} \cos \theta^2 - m^3 \right) \right) \delta \theta^2 \]
\[ + \left( -m^2 g \frac{l^3}{2} \cos \theta^3 + F^3 l^2 \sin \theta^2 \cos \phi \right) \delta \theta^3 \]

\[ \delta W_i = \left[ m^2 \ddot{R}_x \quad m^2 \ddot{R}_y \right] \left[ -\frac{l^2}{2} \sin \theta^2 \\ \frac{l^2}{2} \cos \theta^2 \right] \delta \theta^2 + F^2 \ddot{\theta} \delta \theta^2 \]
\[ + \left[ m^3 \ddot{R}_x \quad m^3 \ddot{R}_y \right] \left[ -l^2 \sin \theta^2 \\ l^2 \cos \theta^2 \right] \left[ \frac{l^3}{2} \sin \theta^3 \\ \frac{l^3}{2} \cos \theta^3 \right] \delta \theta^3 + J^3 \ddot{\theta}^3 \delta \theta^3 \]
Since,
\[ \sum_{i=1}^{n_b} \delta W^i_e - \sum_{i=1}^{n_b} \delta W^i_i = 0 \]

The equations of motion become,
\[
\left( M^2 - m^2 g \frac{l^2}{2} \cos \theta^2 - m^3 g l^2 \cos \theta^2 - F^3 l^2 \sin \theta^2 \cos \phi + F^3 l^2 \cos \theta^2 \sin \phi \right) \\
- \left( m^2 \ddot{x} - m^2 \ddot{y} \right) \left\{ -\frac{l^2}{2} \sin \theta^2 \right\} + J^2 \ddot{\phi} + \left[ m^3 \ddot{R}_x^3 - m^3 \ddot{R}_y^3 \right] \left\{ -\frac{l^2}{2} \sin \theta^2 \right\} = 0
\]
\[
\left( -m^3 g \frac{l^3}{2} \cos \theta^3 - F^3 l^3 \sin \theta^3 \cos \phi + F^3 l^3 \cos \theta^3 \sin \phi \right) - \left[ m^3 \ddot{R}_x^3 - m^3 \ddot{R}_y^3 \right] \left\{ \frac{l^3}{2} \sin \theta^3 \right\} + J^3 \ddot{\phi} = 0
\]

Step 5: Substitute Acceleration Equations into the Equations of Motion
Not included symbolically

Homework
Derive the dynamic equations for this two-link manipulator.

Notes:
- Gravity is active in the negative Y direction
- There is a friction between links 1 and 2
Example:
Derive the equations of the dynamics of the crank slider machine using the principle of virtual work. *Neglect gravity.*

Step 1: Develop Virtual Work Equations
The virtual work of the external forces is,

\[ \delta W_e = M^2 \delta \theta^2 + F^T \cdot \delta R^4 \]

Since \( F^4 \) and is in the same direction as \( R^4_x \),

\[ \delta W_e = M^2 \delta \theta^2 + F^4 \cdot \delta R^4_x \]

Similarly, the virtual work of the inertial forces can be expressed as,

\[ \delta W_i = \left[ m^2 \ddot{R}_x \quad m^2 \ddot{R}_y \right] \delta R^2 + J^2 \ddot{\theta} \delta \theta^2 + \left[ m^3 \ddot{R}_x^3 \quad m^3 \ddot{R}_y^3 \right] \delta R^3 + J^3 \dddot{\theta} \delta \theta^3 + m^4 \dddot{R}_x^4 \delta R^4_x \]

Equating the two virtual works,

\[ M^2 \delta \theta^2 + F^4 \cdot \delta R^4_x = \left[ m^2 \ddot{R}_x \quad m^2 \ddot{R}_y \right] \delta R^2 + J^2 \ddot{\theta} \delta \theta^2 + \left[ m^3 \ddot{R}_x^3 \quad m^3 \ddot{R}_y^3 \right] \delta R^3 + J^3 \dddot{\theta} \delta \theta^3 + m^4 \dddot{R}_x^4 \delta R^4_x \]
Step 2: Develop Virtual Displacement Equations

From kinematics, the constraint equations are,

\[
C(q, t) = \begin{bmatrix}
R^1_1 \\
\theta^1 \\
R^1_1 - R^2 - A^2 \bar{u}^2_0 \\
R^2 + A^2 \bar{u}^2_A - R^3 - A^3 \bar{u}^3_A \\
R^3 + A^3 \bar{u}^3_B - R^4 \\
R^4_y \\
\theta^4
\end{bmatrix} = 0
\]

The Jacobian matrix is,

\[
C_q = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This matrix can be rearranged and used to solve for virtual displacements in terms of \( \delta \theta^2 \)

\[
\delta q = \begin{bmatrix}
-C_q^{-1} C_{q_i}
\end{bmatrix} \delta q_i = B_i \delta q_i
\]

This means we need to express all variations in terms of \( \delta \theta^2 \).

Step 3: Develop Acceleration Equations

The acceleration terms can be obtained from the kinematic analysis of Chapter 3 using this equation:

\[
C_q \ddot{q} = -(C_q \dot{q}) \ddot{q} - 2C_q \dot{q} \dot{q} - C_{tt} = Q_d
\]

Since,

\[
C_q \dot{q} = C_{tt} = 0 \\
\dot{q} = C_q^{-1} \left(-(C_q \dot{q}) \dot{q} \right)
\]
Step 4: Substitute Virtual Displacement Equations into the Virtual Work Equation
The result will be,
\[
(M^2 - F^4 \frac{l^2 \sin(\theta^2 - \theta^3)}{\cos \theta^3})
\]
\[
= \left[ m^2 \ddot{R}_x \quad m^2 \ddot{R}_y \right] \left\{ \frac{-l^2 \sin \theta^2}{2} \quad \frac{l^2 \cos \theta^2}{2} \right\} + J^2 \ddot{\theta}^2
\]
\[
+ \left[ m^3 \ddot{R}_x \quad m^3 \ddot{R}_y \right] \left\{ -l^2 \sin \theta^2 \quad \frac{l^3 \sin \theta^3}{2} \right\} + \left\{ \frac{-l^3 \sin \theta^3}{2} \quad \frac{-l^2 \cos \theta^2}{l^3 \cos \theta^3} \right\} + J^3 \ddot{\theta}^3 \frac{-l^2 \cos \theta^2}{l^3 \cos \theta^3}
\]
\[
+ m^4 \ddot{R}_x \left( -l^2 \sin \theta^2 - l^3 \sin \theta^3 \frac{l^2 \cos \theta^2}{l^3 \cos \theta^3} \right)
\]

Step 5: Substitute Acceleration Equations into the Equations of Motion
Not included symbolically

**Homework**
Derive the dynamic equations for this machine.

*Note: Gravity is active in the negative Y direction*
Hamiltonian Equations

We know from D’Alembert’s Principle that,
\[ F^i - m^i \dot{a}^i = 0 \]
*Newton’s Equations*

If \( F^i \) is selected to be at the center of mass of the body, the above equations can be multiplied by a virtual displacement without affecting it,
\[ (F^i - m^i \ddot{a}^i) \delta r^i = 0 \]

where, \( R^i \) is the global position vector of the center of mass of body \( i \).

The second term of the above equation \( m^i \ddot{R}^i \) needs a closer consideration.

It is easy to show that,
\[
\frac{d}{dt}((\ddot{R}^i)^T \delta R^i) = (\ddot{R}^i)^T \delta R^i + (\dot{R}^i)^T \delta \dot{R}^i \\
\frac{d}{dt}((\dot{R}^i)^T \delta R^i) = (\dot{R}^i)^T \delta r^i + \frac{1}{2} \delta ((\dot{R}^i)^T \dot{R}^i)
\]

We know that the kinetic energy of body \( i \) is,
\[ T^i = \frac{1}{2} m^i (\dot{R}^i)^T \dot{R}^i \]

Therefore,
\[ \delta T^i = \frac{1}{2} \delta (m^i (\dot{R}^i)^T \dot{R}^i) \]

Multiplying by \( m^i \) and substituting in the variational equation
\[ m^i \frac{d}{dt}((\dot{R}^i)^T \delta R^i) = m^i (\dot{R}^i)^T \delta R^i + \delta T^i \]

Or,
\[ -m^i (\ddot{R}^i)^T \delta R^i = \delta T^i - m^i \frac{d}{dt}((\dot{R}^i)^T \delta R^i) \]
D’Alembert’s equation becomes,

\[ F^T_i \delta R^i - m^i \frac{d}{dt} \left( (\dot{R}^i)^T \delta R^i \right) + \delta T^i = 0 \]

Rearranging,

\[ \delta W_e^i + \delta T^i = m^i \frac{d}{dt} \left( (\dot{R}^i)^T \delta R^i \right) \]

where \( W^i \) is the work of the external forces.

If the above equation is summed for all the bodies in the system and integrated over time between two time instants where all the states of the system are known. **Examples are initial conditions and final state of the system at \( t=\infty \),**

\[ \int_{t_1}^{t_2} (\delta W_e + \delta T) \, dt = \sum_{i=1}^{n_b} \left( \int_{t_1}^{t_2} \left( m^i \frac{d}{dt} \left( (\dot{R}^i)^T \delta R^i \right) \right) \, dt \right) \]

The integrand and differential operator will cancel each other.

\[ \int_{t_1}^{t_2} (\delta W_e + \delta T) \, dt = \sum_{i=1}^{n_b} \left[ m^i \left( (\dot{R}^i)^T \delta R^i \right) \right]_{t_1}^{t_2} \]

\[ \int_{t_1}^{t_2} (\delta W_e + \delta T) \, dt - \sum_{i=1}^{n_b} \frac{\partial T}{\partial R^i} \delta R^i \bigg|_{t_1}^{t_2} = 0 \]

The above equation is labeled **Hamilton Principle.**

If \( R^i \) is known at \( t_1 \) and \( t_2 \), its variation, \( \delta R^i = 0 \). Therefore,

\[ \int_{t_1}^{t_2} (\delta W_e + \delta T) \, dt = 0 \]
Remember that the forces acting on a mechanical system are:

- **Conservative forces**: $Q_{co}$. Conservative forces can be obtained using a potential function. Example is a gravity field ($mgh$) or a linear spring.
- **Nonconservative forces**: $Q_{nc}$. Examples are the damping, friction, and actuator forces.

$$Q_e = Q_{co} + Q_{nc}$$

A conservative force can be expressed as,

$$Q_{co} = - (\frac{\partial V}{\partial q})^T$$

where, $V$ is the potential energy function.

Therefore, the Hamilton Principle can be then expressed as,

$$\int_{t_1}^{t_2} (-\delta V + \delta T + \delta W_{nc}) dt = 0$$

Hamilton Principle is equivalent to Newton second law except that it is in terms of energies.

For a conservative system,

$$\int_{t_1}^{t_2} (-\delta V + \delta T) dt = 0$$

Define the Lagrangian as,

$$L = T - V$$

$$\int_{t_1}^{t_2} \delta L dt = 0$$

Or,

$$\delta \int_{t_1}^{t_2} L dt = 0$$

This equation is referred to also Principle of Least Action.

This principle represents an interesting way to understand dynamics.

It means that the movement of any system will move in a unique way that will make the variation of the integral of the difference between kinetic and potential energies between any two time instants equal to zero.

Another way to think of the dynamics of any system is that it always moves while satisfying that $\int_{t_1}^{t_2} L dt$ is at minimum.
Example:
Derive the dynamic equations for this cylinder that rolls without slipping on a curved surface using the Hamiltonian formulation.

Note: Gravity is active in the negative Y direction

Step 1: Develop Kinematic Equations
This is a new kinematic constraint (disk rolling on curved surface).

\( \theta^2 \) is the independent coordinate of the system.

We can easily see that,

\[ \theta^2 r^2 = \phi (r^1 - r^2) \]

Therefore at any instant,

\[
R^2_x = R^1_x - (r^1 - r^2) \sin(\phi) = R^1_x - (r^1 - r^2) \sin \left( \frac{\theta^2 r^2}{r^1 - r^2} \right)
\]

\[
R^2_y = R^1_y - (r^1 - r^2) \cos(\phi) = R^1_y - (r^1 - r^2) \left( \cos \left( \frac{\theta^2 r^2}{r^1 - r^2} \right) \right)
\]

The kinematic constraints of the system are,

\[
\mathcal{C} = \begin{cases} 
R^1 \\
\theta^1 \\
-R^1 + R^2 + (r^1 - r^2) \begin{bmatrix} sin \left( \frac{\theta^2 r^2}{r^1 - r^2} \right) \\
-\cos \left( \frac{\theta^2 r^2}{r^1 - r^2} \right) \end{bmatrix} 
\end{cases}
\]
The Jacobian is,

\[
C_q = \begin{bmatrix}
I & 0 & 0 \\
-1 & I & 0 \\
0 & -1 & I \\
\end{bmatrix}
\begin{bmatrix}
0 \\
r^2 \\
C_{qi} \\
\end{bmatrix}
\begin{bmatrix}
\cos\left(\frac{\theta^2 r^2}{r^1 - r^2}\right) \\
\sin\left(\frac{\theta^2 r^2}{r^1 - r^2}\right) \\
1 \\
\end{bmatrix}
\]

This matrix can be rearranged and used to solve for virtual displacements in terms of \(\delta \theta^2\)

\[
\dot{q} = \left\{-C_{qd}^{-1}C_{qi}\right\} \dot{q}_i = B_i \dot{q}_i
\]

where,

\[
C_{qd} = \begin{bmatrix}
I & 0 & 0 \\
-1 & 0 & 0 \\
0 & 0 & I \\
\end{bmatrix}
\]

\[
\dot{q} = B_i \dot{q}_i = \left\{-C_{qd}^{-1}C_{qi}\right\} \dot{q}_i
\]

Typically, we do not pursue complete solution as long as we have the basic matrices. We will however do so here to understand the problem.

\[
\dot{q} = \begin{bmatrix}
0 \\
0 \\
0 \\
r^2 \left(\cos\left(\frac{\theta^2 r^2}{r^1 - r^2}\right)\right) \\
r^2 \left(\sin\left(\frac{\theta^2 r^2}{r^1 - r^2}\right)\right) \\
1 \\
\end{bmatrix}
\]
**Step 2: Develop Kinetic Energy Equation**

The kinetic energy of the cylinder is,

\[ T = \frac{1}{2} m^2 \left( (\dot{R}_x^2)^2 + (\dot{R}_y^2)^2 \right) + \frac{1}{2} J^2 (\dot{\theta}^2)^2 \]

The second term accounts for the rolling of the cylinder around its axis

Using \( B_i \) equation, the kinetic energy becomes,

\[ T = \frac{1}{2} m^2 \left( (B_{i4.1} \dot{\theta}^2)^2 + (B_{i5.1} \dot{\theta}^2)^2 \right) + \frac{1}{2} J^2 (\dot{\theta}^2)^2 \]

Since mass moment of inertia of a disk is,

\[ J^2 = \frac{m^2 (r^2)^2}{2} \]

\[ T = \frac{1}{2} m^2 (r^2)^2 (\dot{\theta}^2)^2 + \frac{1}{2} \frac{m^2 (r^2)^2}{2} (\dot{\theta}^2)^2 = \frac{3}{4} m^2 (r^2)^2 (\dot{\theta}^2)^2 \]

**Step 3: Develop Potential Energy Equation**

The potential energy of the cylinder:

\[ V = m^2 g (R_y^2 - R_{y0}^2) \]

From the kinematic constraint equations, it can be shown that,

\[ R_y^2 = R_{1y}^2 - (r^1 - r^2) \left( \cos \left( \frac{\theta^2 r^2}{r^1 - r^2} \right) \right) \]

The system configuration at \( t = \infty \) is *(why?)*,

\[ R_{y0}^2 = -(r^1 - r^2) \]

Therefore,

\[ V = m^2 g \left( R_{1y}^2 + (r^1 - r^2) \left( 1 - \cos \left( \frac{\theta^2 r^2}{r^1 - r^2} \right) \right) \right) \]

**Step 4: Combine Kinetic and Potential Energies to form the Lagrangian**

The Lagrangian is given by:

\[ L = T - V \]

\[ L = \frac{3}{4} m^2 (r^2)^2 (\dot{\theta}^2)^2 - m^2 g \left( R_{1y}^2 + (r^1 - r^2) \left( 1 - \cos \left( \frac{\theta^2 r^2}{r^1 - r^2} \right) \right) \right) \]

At this stage, we will get rid of \( R_{1y} \). Therefore,

\[ L = \frac{3}{4} m^2 (r^2)^2 (\dot{\theta}^2)^2 - m^2 g \left( (r^1 - r^2) \left( 1 - \cos \left( \frac{\theta^2 r^2}{r^1 - r^2} \right) \right) \right) \]
Step 5: Obtain the Variation of the Lagrangian

The variation of the Lagrangian is,

$$\delta L = \frac{\partial L}{\partial \theta^2} \delta \theta^2 + \frac{\partial L}{\partial \dot{\theta}^2} \delta \dot{\theta}^2$$

Simplifying,

$$\delta L = \left( m^2 g (r^1 - r^2) \left( \frac{r^2}{r^1 - r^2} \right) \left( \sin \left( \frac{\theta^2 r^2}{r^1 - r^2} \right) \right) \right) \delta \theta^2 + \left( \frac{3}{2} m^2 (r^2)^2 (\dot{\theta}^2) \right) \delta \dot{\theta}^2$$

Here we have to interrupt the solution to learn a technique that can enable us to express $\delta \dot{\theta}$ in terms of $\delta \theta$ using integration by parts. This technique is variation on differentiation by parts.

Proof:

We know that,

$$\frac{d}{dt} (AB) = A \frac{d}{dt} (B) + B \frac{d}{dt} (A)$$

Rearranging,

$$A \frac{d}{dt} (B) = \frac{d}{dt} (AB) - B \frac{d}{dt} (A)$$

Integrating both sides between the same time instants will not disturb the above equation. Therefore,

$$\int_{t_1}^{t_2} A \frac{d}{dt} (B) \, dt = \int_{t_1}^{t_2} \frac{d}{dt} (AB) \, dt - \int_{t_1}^{t_2} B \frac{d}{dt} (A) \, dt$$

$$\int_{t_1}^{t_2} A \frac{d}{dt} (B) \, dt = (AB)|_{t_1}^{t_2} - \int_{t_1}^{t_2} B \frac{d}{dt} (A) \, dt$$
Step 6: Integrate the Variation of the Lagrangian between two time instants

If we apply the integration by parts principle to second part of \( \int_{t_1}^{t_2} \delta L \, dt \),

\[
\int_{t_1}^{t_2} \left( \frac{3}{2} m^2 (r^2)^2 \dot{\theta}^2 \right) \delta \dot{\theta}^2 \, dt
= \frac{3}{2} m^2 (r^2)^2 \int_{t_1}^{t_2} \left( \frac{d}{dt} (\dot{\theta}^2) \right) (\delta \dot{\theta}^2) \, dt
= \frac{3}{2} m^2 (r^2)^2 \left( \int_{t_1}^{t_2} \frac{d}{dt} (\dot{\theta}^2 \delta \theta^2) \, dt - \int_{t_1}^{t_2} (\dot{\theta}^2) \delta \theta^2 \, dt \right)
\]

The first term on the right side vanishes because it is assumed that everything is known at times \( t_1 \) and \( t_2 \). The integral of \( \delta L \) becomes

\[
\int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \left( m^2 g (r^2) \left( \frac{\dot{\theta}^2 r^2}{r_1 - r^2} \right) \right) \delta \theta^2 - \left( \frac{3}{2} m^2 (r^2)^2 \ddot{\theta}^2 \right) \delta \theta^2 \, dt
\]

Since \( \delta \theta^2 \) cannot vanish, the equation of motion is,

\[
m^2 g (r^2) \left( \frac{\dot{\theta}^2 r^2}{r_1 - r^2} \right) - \left( \frac{3}{2} m^2 (r^2)^2 \ddot{\theta}^2 \right) = 0
\]

Simplifying,

\[
g \left( \frac{\dot{\theta}^2 r^2}{r_1 - r^2} \right) + \frac{3}{2} r^2 \ddot{\theta}^2 = 0
\]

Or,

\[
\ddot{\theta}^2 = - \frac{2g}{3r^2} \sin \left( \frac{\dot{\theta}^2 r^2}{r_1 - r^2} \right)
\]

This equation can be simplified for small angle oscillations,

\[
\ddot{\theta}^2 = - \frac{2g}{3} \left( \frac{1}{r_1 - r^2} \right) \theta^2
\]
Homework
Derive the dynamic equations for this two-link manipulator using Hamiltonian Equations.

Notes:
- Gravity is active in the negative Y direction
- There is no friction between links 1 and 2
Lagrange’s Equations

Joseph-Louis Lagrange (1736-1813)
(Source Wikipedia)

- Hamilton’s Principle can be a tedious process for complicated systems.
- The principle of virtual work is a basis for an approach that can be used to derive the equations of a dynamic system.
Based on Hamilton’s Principle,
\[ \int_{t_1}^{t_2} (-\delta V + \delta T + \delta W_{nc}) \, dt = 0 \]

Or,
\[ \int_{t_1}^{t_2} \delta L \, dt + \int_{t_1}^{t_2} \delta W_{nc} \, dt = 0 \]

The variation of \( L \) is,
\[ \delta L = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \]

Similarly, the variation of the nonconservative work is,
\[ \delta W_{nc} = Q_{nc} \delta q \]

The second term of the Lagrangian variation can be expressed as using integration by parts,
\[ \int_{t_1}^{t_2} \frac{d}{dt} \left( B \frac{\partial}{\partial \dot{q}} \left( T - V \right) \right) \delta q \, dt = \int_{t_1}^{t_2} \frac{d}{dt} \left( AB \right) \delta q \, dt - \int_{t_1}^{t_2} B \frac{d}{dt} \left( A \right) \delta q \, dt \]

In this case,
\[ \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta q \, dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial q} \delta \dot{q} \, dt = \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \left|_{t_1}^{t_2} \right. - \left. \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \, dt \right. \]

Since variations of \( q \) vanish at \( t_1 \) and \( t_2 \), the first term on the right-hand side is equal to zero.
\[ \int_{t_1}^{t_2} \left( -\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} + Q_{nc} \right) \delta q \, dt = 0 \]

Since \( \delta q \neq 0 \), we end with a system of equations of motion for the generalized coordinates can be expressed as the Lagrange’s equations of motion,
\[ \frac{d}{dt} \left( \frac{\partial (T - V)}{\partial \dot{q}} \right)^T - \left( \frac{\partial (T - V)}{\partial q} \right)^T = Q_{nc} \]

or,
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_{ncj} \quad j = 1, 2, \ldots, n \]
Example:
Derive the dynamic equations for this two-link manipulator using the Lagrangian approach.

Note: Gravity is active in the negative Y direction

\( \theta^2 \) and \( \theta^3 \) are the independent coordinates of the system.

Before proceeding to solve the problem, we start by obtaining the kinematic relationships of the system.

**Step 1: Obtain Kinematic Equations of the System**

As done earlier, we use the standardize the process of obtaining position and velocity equations, first in terms of the generalized coordinates then in terms of the independent coordinates:

\[
C = \begin{bmatrix}
    R^2 + A^2 \left\{ -\frac{l^2}{2} \right\} & 0 \\
    -R^2 - A^2 \left\{ \frac{l^2}{2} \right\} + R^3 - A^3 \left\{ -\frac{l^3}{2} \right\} & 0 \\
    0 & 0 \\
    \end{bmatrix}
\]

\[
C_q = \begin{bmatrix}
    I & A^2 \theta \left\{ -\frac{l^2}{2} \right\} & 0 & 0 \\
    0 & 0 & I & A^3 \theta \left\{ -\frac{l^3}{2} \right\} \\
    \end{bmatrix}
\]

\[
C_{qd} = \begin{bmatrix}
    I & 0 \\
    0 & I \\
    \end{bmatrix}
\]

\[
C_{qi} = \begin{bmatrix}
    A^2 \theta \left\{ -\frac{l^2}{2} \right\} & 0 \\
    A^2 \theta \left\{ \frac{l^2}{2} \right\} & A^3 \theta \left\{ -\frac{l^3}{2} \right\} \\
    \end{bmatrix}
\]

\[
\dot{q} = B_i \ddot{q}_i = \begin{bmatrix}
    -C_{qi}^{-1}C_{qi} \\
    I \\
    \end{bmatrix} \ddot{q}_i
\[
\dot{q} = \begin{bmatrix}
-l^2 \sin \theta^2 / 2 & 0 \\
 l^2 \cos \theta^2 / 2 & 0 \\
-l^2 \sin \theta^2 / 2 & -l^3 \sin \theta^3 / 2 \\
 l^2 \cos \theta^2 & l^3 \cos \theta^3 / 2 \\
 1 & 0 \\
 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}^2 \\
\dot{\theta}^3
\end{bmatrix}
\]

The same analysis can be extended to the tip point, which is needed because of the force at the tip:

\[
r^3_p = R^3 + A^3 \begin{bmatrix} l^3 / 2 \\
 0 \end{bmatrix}
\]

\[
\delta r^3_p = \delta R^3 + A^3 \theta \begin{bmatrix} l^3 / 2 \\
 0 \end{bmatrix} \delta \theta^3
\]

\(\delta R^3\) can be obtained for the \(\dot{q}\) equation (3\(^{rd}\) and 4\(^{th}\) rows). Therefore,

\[
\delta r^3_p = \begin{bmatrix}
-l^2 \sin \theta^2 & -l^3 \sin \theta^3 \\
 l^2 \cos \theta^2 & l^3 \cos \theta^3
\end{bmatrix}
\begin{bmatrix}
\delta \theta^2 \\
\delta \theta^3
\end{bmatrix}
\]
Step 2: Derive Kinetic Energy Equations of the System
The above equations can be used to determine velocities. Therefore, the kinetic energy of the manipulator is,

\[ T = T^2 + T^3 \]

\[ T^i = \frac{1}{2} m^2 \left[ \ddot{R}_x \dot{R}_y - \ddot{R}_y \dot{R}_x \right] + \frac{1}{2} J^i (\dot{\theta}^i)^2 \]

Replacing \( \ddot{R}_x \) and \( \ddot{R}_y \) by \( B_i \) and \( \dot{q}_i \),

\[ T^2 = \frac{1}{2} m^2 \left[ -\frac{l^2}{2} \dot{\theta}^2 \cos \theta^2 - \frac{l^2}{2} \dot{\theta}^2 \sin \theta^2 \right] + \frac{1}{2} J^2 (\dot{\theta}^2)^2 \]

\[ T^3 = \frac{1}{2} m^3 \left[ -\frac{l^2}{2} \dot{\theta}^2 \cos \theta^2 - \frac{l^2}{2} \dot{\theta}^2 \sin \theta^2 \right] + \frac{1}{2} J^3 (\dot{\theta}^3)^2 \]

Simplifying,

\[ T = \frac{1}{2} m^2 \left( \frac{l^2}{2} \dot{\theta}^2 \right)^2 + \frac{1}{2} J^2 (\dot{\theta}^2)^2 + \frac{1}{2} m^3 \left( \frac{l^2}{2} \dot{\theta}^2 \right)^2 + \frac{1}{2} m^3 \left( \frac{l^3}{2} \dot{\theta}^3 \right)^2 + 2 \left( \frac{l^2}{2} \dot{\theta}^2 \right) \left( \frac{l^3}{2} \dot{\theta}^3 \right) \cos (\theta^3 - \theta^2) + \frac{1}{2} J^3 (\dot{\theta}^3)^2 \]

Step 3: Derive Potential Energy Equations of the System
The potential energy is related to the gravity field and can be expressed as,

\[ V = m^2 g R^2_y + m^3 g R^3_y \]

\( R^2_y \) and \( R^3_y \) can be easily obtained from \( C \). Substituting,

\[ V = m^2 g \left( \frac{l^2}{2} \sin \theta^2 \right) + m^3 g \left( l^2 \sin \theta^2 + \frac{l^3}{2} \sin \theta^3 \right) \]

Step 4: Derive Virtual Work of the Non-Conservative Forces of the System
The work equation of the external forces is:

\[ W_{nc} = M^2 \dot{\theta}^2 + F^3 T r^3_p \]

The corresponding virtual work is,

\[ \delta W_{nc} = M^2 \dot{\theta}^2 + F^3 T \delta r^3_p \]

Substituting \( \delta r^3_p \) in the virtual work equations:

\[ \delta W_{nc} = M^2 \dot{\theta}^2 + [F^3 \cos \phi - F^3 \sin \phi] \left[ \frac{-l^2 \sin \theta^2}{l^2 \cos \theta^2} - \frac{l^3 \sin \theta^3}{l^3 \cos \theta^3} \right] \{ \dot{\theta}^2 \} \]

Rearranging,

\[ \delta W_{nc} = (M^2 - F^3 l^2 \sin \theta^2 \cos \phi + F^3 l^2 \cos \theta^2 \sin \phi) \dot{\theta}^2 \\
+ (-F^3 l^3 \sin \theta^3 \cos \phi + F^3 l^3 \cos \theta^3 \sin \phi) \delta \theta^3 \]

Therefore,

\[ Q_{\theta^2} = M^2 - F^3 l^2 \sin \theta^2 \cos \phi + F^3 l^2 \cos \theta^2 \sin \phi \]

\[ Q_{\theta^3} = -F^3 l^3 \sin \theta^3 \cos \phi + F^3 l^3 \cos \theta^3 \sin \phi \]
Step 5: Arrange the Lagrange Equations of the System

The Lagrange equations of motion

\[
\frac{d}{dt} \left( \frac{\partial (T - V)}{\partial \dot{\theta}^2} \right) - \frac{\partial (T - V)}{\partial \theta^2} = Q_{\dot{\theta}^2}
\]

\[
\frac{d}{dt} \left( \frac{\partial (T - V)}{\partial \dot{\theta}^3} \right) - \frac{\partial (T - V)}{\partial \theta^3} = Q_{\dot{\theta}^3}
\]

We will list each term separately,

\[
\frac{\partial (T - V)}{\partial \dot{\theta}^2} = \left( m^2 \left( \frac{l^2}{2} \right)^2 \dot{\theta}^2 + l^2 \dot{\theta}^2 + m^3 \left( \left( \frac{l^2}{2} \right)^2 \dot{\theta}^2 + l^2 \left( \frac{l^3}{2} \dot{\theta}^3 \right) \cos(\theta^3 - \theta^2) \right) \right)
\]

\[
\frac{\partial (T - V)}{\partial \theta^2} = + \frac{1}{2} m^3 \left( 2 \left( \frac{l^2}{2} \dot{\theta}^2 \right) \left( \frac{l^3}{2} \dot{\theta}^3 \right) \sin(\theta^3 - \theta^2) \right) - \left( m^2 g \frac{l^2}{2} \cos \theta^2 + m^3 g (l^2 \cos \theta^2) \right)
\]

\[
\frac{\partial (T - V)}{\partial \dot{\theta}^3} = m^3 \left( \left( \frac{l^3}{2} \right)^2 \dot{\theta}^3 + \left( \frac{l^2}{2} \right)^2 \dot{\theta}^2 \right) \cos(\theta^3 - \theta^2)
\]

\[
\frac{\partial (T - V)}{\partial \theta^3} = - \frac{1}{2} m^3 \left( 2 \left( \frac{l^2}{2} \dot{\theta}^2 \right) \left( \frac{l^3}{2} \dot{\theta}^3 \right) \sin(\theta^3 - \theta^2) \right) - \left( m^3 g \frac{l^3}{2} \cos \theta^3 \right)
\]

\[Q_{\dot{\theta}^2} = M^2 - F^3 l^2 \sin \theta^2 \cos \phi + F^3 l^2 \cos \theta^2 \sin \phi\]

\[Q_{\dot{\theta}^3} = - F^3 l^3 \sin \theta^3 \cos \phi + F^3 l^3 \cos \theta^3 \sin \phi\]
The equations of motion can be assembled as,
\[
\frac{d}{dt} \left( m^2 \left( \frac{l^2}{2} \right)^2 \dot{\theta}^2 + l^2 \dot{\theta}^2 + m^3 \left( (l^2)^2 \dot{\theta}^2 + l^2 \left( \frac{l^3}{2} \dot{\theta}^3 \right) \cos(\theta^3 - \theta^2) \right) \right)
\]
\[
- \left( \frac{1}{2} m^3 \left( 2l^2 \dot{\theta}^2 \left( \frac{l^3}{2} \dot{\theta}^3 \right) \sin(\theta^3 - \theta^2) \right) - \left( m^2 g \frac{l^2}{2} \cos \theta^2 + m^3 g(l^2 \cos \theta^2) \right) \right)
\]
\[
= M^2 - F^3 l^2 \sin \theta^2 \cos \theta + F^3 l^2 \cos \theta^2 \sin \theta
\]
\[
\frac{d}{dt} \left( m^3 \left( \frac{l^3}{2} \right)^2 \dot{\theta}^3 + (l^2 \dot{\theta}^2) \left( \frac{l^3}{2} \right) \cos(\theta^3 - \theta^2) \right)
\]
\[
- \left( -\frac{1}{2} m^3 \left( 2l^2 \dot{\theta}^2 \left( \frac{l^3}{2} \dot{\theta}^3 \right) \sin(\theta^3 - \theta^2) \right) - \left( m^3 g \frac{l^3}{2} \cos \theta^3 \right) \right)
\]
\[
= -F^3 l^3 \sin \theta^3 \cos \theta + F^3 l^3 \cos \theta^3 \sin \theta
\]

Differentiating with respect to time and rearranging, we can reach the following equations.
\[
m^2 \left( \frac{l^2}{2} \right)^2 \ddot{\theta}^2 + l^2 \ddot{\theta}^2 + m^3 \left( (l^2)^2 \ddot{\theta}^2 + l^2 \left( \frac{l^3}{2} \ddot{\theta}^3 \right) \cos(\theta^3 - \theta^2) \right)
\]
\[
- \left( \frac{1}{2} m^3 \left( 2l^2 \ddot{\theta}^2 \left( \frac{l^3}{2} \ddot{\theta}^3 \right) \sin(\theta^3 - \theta^2) \right) - \left( m^2 g \frac{l^2}{2} \cos \theta^2 + m^3 g(l^2 \cos \theta^2) \right) \right)
\]
\[
= M^2 - m^2 g \frac{l^2}{2} \cos \theta^2 - m^3 g l^2 \cos \theta^2 - F^3 l^2 \sin \theta^2 \cos \theta + F^3 l^2 \cos \theta^2 \sin \theta
\]
\[
m^3 \left( \frac{l^3}{2} \right)^2 \ddot{\theta}^3 + (l^2 \ddot{\theta}^2) \left( \frac{l^3}{2} \right) \cos(\theta^3 - \theta^2) - \left( \frac{l^3}{2} \ddot{\theta}^3 \left( 2\ddot{\theta}^2 \ddot{\theta}^3 - (\ddot{\theta}^2)^2 \right) \sin(\theta^3 - \theta^2) \right)
\]
\[
= -m^3 g \frac{l^3}{2} \cos \theta^3 - F^3 l^3 \sin \theta^3 \cos \theta + F^3 l^3 \cos \theta^3 \sin \theta
\]

Rearranging in matrix form,
\[
\begin{bmatrix}
    m^2 \left( \frac{l^2}{2} \right)^2 + l^2 + m^3((l^2)^2) & m^3 \left( \frac{l^3}{2} \right) \cos(\theta^3 - \theta^2) & m^3 \left( \frac{l^3}{2} \right)^2
    \\
    m^3 \left( \frac{l^3}{2} \right) \cos(\theta^3 - \theta^2) & m^3 \left( \frac{l^3}{2} \right)^2 & m^3 \left( \frac{l^3}{2} \right)^2
    \\
\end{bmatrix}
\begin{bmatrix}
    \ddot{\theta}^2 \\
    \ddot{\theta}^3 \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
    M^2 - m^2 g \frac{l^2}{2} \cos \theta^2 - m^3 g l^2 \cos \theta^2 - F^3 l^2 \sin \theta^2 \cos \theta + F^3 l^2 \cos \theta^2 \sin \theta \\
    -m^3 g \frac{l^3}{2} \cos \theta^3 - F^3 l^3 \sin \theta^3 \cos \theta + F^3 l^3 \cos \theta^3 \sin \theta \\
\end{bmatrix}
\]

This is equation is similar to what can be obtained using the Embedding Technique:
\[
[B_l^T M B_l] \ddot{q}_i = B_l^T Q_e - B_l^T M \gamma_i
\]
Homework
Derive the dynamic equations for this two-link manipulator using Lagrange’s approach.

Notes:
- Gravity is active in the negative Y direction
- There is no friction between links 1 and 2
Summary of Chapters 4 and 5:

**Free Body Diagrams**

**Advantages:**
- Simple and consistent

**Disadvantages:**
- The Generalized coordinates and constraints forces are mixed in the equations of motion.
- This however can be avoided if Embedding Technique is used

**Virtual Work**

**Advantages:**
- Eliminate the need to consider constraint forces while deriving equations of motion

**Disadvantages:**
- Two steps are needed: Virtual displacement and Virtual Work

**Hamilton Principle**

**Advantages:**
- Energy-based
- One-step approach

**Disadvantages:**
- Complicated
- Integral-based

**Lagrange’s Equations**

**Advantages:**
- Energy-based
- One-step approach
- Simple

**Disadvantages:**
- Simplicity is reduced if the system is non-conservative